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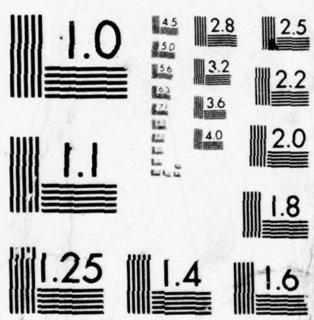

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THERMAL REGIME OF OIL AND GAS WELLS

E.A. Bondarev and B.A. Krasovitskiy

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The book examines questions involving the calculation of thermal regimes of oil and gas flows in a borehole, and also involving the temperature fields in the adjoining rock. Special attention is given to heat exchange between wells and frozen ground under various regimes of boring and exploitation. A methodology is given for calculating the configuration of the thawing front around the well. The work is of interest to engineers and scientific workers who are concerned with questions involving the development of oil and gas deposits.

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FOREWORD

The increase in oil and gas production has led inescapably to the necessity of increasing the accuracy of calculations made in designing methods for the development of oil and gas deposits and of increasing the reliability of prospecting hole tests. This in turn requires a deeper and more detailed description of the processes and phenomena which take place during the operation of oil and gas wells. Until recently, complications connected with this served as an obstacle to a more precise and rigorous formulation of the appropriate problems. With the advent of computers and the development of numerical methods of integration of the equations of mathematical physics, this obstacle may to a significant degree be overcome.

In the present work the authors primarily strive to illuminate several new questions involving the nonisothermal movement of fluid and gas in pipes, namely, the processes of the thermal interaction of oil and gas wells with frozen rock. In addition, the authors wish to generalize and systematize several applied aspects of pipe hydraulics.

The work was carried out at the I.M. Gubkin Moscow Institute of Petrochemical and Gas Industry and at the Laboratory of the Mechanics Continua and Dispersed Media of the Institute of Physical and Technical

Problems of the North, affiliated with the Yakutsk Branch of the SO AN SSSR.

The authors express their thanks to their senior scientific coworker at the Department of Petroleum & Gas and Subterranean Hydromechanics, V.I. Maron, for his great assistance in preparing the material of this book, and also to the coworkers of the Laboratory of Continuum and Dispersed Mechanics who assisted and took part in the investigations, especially L.K. Stepanets, who devoted much effort in the layout of this book and in preparing the manuscript for printing.

E.A. Bondarev wrote Section 1 of Chapter 1 and Sections 1-3 of Chapter 2, and B. A. Krasovitskiy wrote Section 2 of Chapter 1, Section 4 of Chapter 2, Chapter 3, and the Appendices.

LIST OF SYMBOLS

$\bar{\Theta}_1$, $\bar{\Theta}_2$ --temperatures of thawed and frozen regions respectively
$\bar{\Theta}_F$ --initial temperature of frozen ground
a --radius of the outer wall of the borehole
$\bar{S}$ --distance from the thawing front to the axis of the borehole
a_1, a_2 --coefficients of thermal diffusivity of thawed and frozen ground respectively
$\bar{\lambda}_1$, $\bar{\lambda}_2$ --coefficients of thermal conductivity of thawed and frozen ground respectively
$\bar{\alpha}$ --coefficients of heat transfer from gas, oil, and flushing fluid to the wall of the borehole
δ_c --thickness of the cement collar of the borehole
λ_c --coefficient of thermal conductivity of cement stone
$\bar{T}$ --temperature of gas, oil in well
ℓ --latent heat of fusion of water
ρ_T --moisture content of the ground by volume
$\bar{t}$ --time
t_0 --time scale
$\bar{t}_m$ --time of beginning of thawing
L --thickness of stratum of permafrost
C_p --specific heat of gas, oil
ρ --density of gas, oil
v --central speed of gas, oil in well
M, G --well output and flushing fluid velocity
p_1 --pressure at entrance to stratum of frozen ground
p_2 --pressure at exit from stratum of frozen ground
δ_w --thickness of wall of pipe
λ_w --coefficient of thermal conductivity of wall of pipe
μ_i --Joule-Thomson coefficient
A --heat equivalent of work
$\bar{T}_1$ --temperature of flushing fluid in pipe
$\bar{T}_2$ --temperature of flushing fluid in annular space
k_{1-2} --coefficient of heat transfer between flushing fluid in drill pipe and in annular space
c --specific heat of flushing fluid
b --diameter of drill pipe
$T_{entrance}$ --temperature of flushing fluid at entrance to drill pipe
z --coefficient of compressibility
ν_ℓ, λ_ℓ --kinematic viscosity and coefficient of thermal conductivity of flushing fluid

CHAPTER 1
REVIEW OF COMPLETED INVESTIGATIONS

The description of the processes which take place during the movement of gas and oil in pipes is based on the use of three fundamental laws: the laws of conservation of mass, momentum, and energy. Analysis of these processes is carried out using the methods of continuum mechanics.

From the point of view of continuum mechanics, a liquid (oil) and a gas are defined as compressible media in which there may exist only negative normal stresses (pressure), and also tangential stresses due to viscous friction. In this case all three conservation laws may be given in the form of several mathematical laws which are generally valid for describing the motion of any continuous medium.

The form of these laws is simplified considerably for one-dimensional motions in pipes having constant cross-sections. Hydraulics is concerned with analyzing such motions. Hydraulics is one of the applied subfields of hydrodynamics; the methods of theoretical investigation of the problems of hydraulics proper are continually approaching the corresponding methods of hydromechanics (methods for solving boundary-layer problems, and theories of turbulence and gas dynamics).

The fundamental results of liquid and gas pipe hydraulics may

be found in the second volume of the survey, "Mechanics in the USSR over 50 years" [1]; therefore, in what follows we shall only consider those investigations which are directly connected with the thermal regime of wells.

Section 1. Investigations of Heat Transfer Processes During the Movement of Oil and Gas in Wells

In spite of the obvious generality of the processes which take place during the movement of a liquid or gas in pipelines and wells, the study of the thermal regime of the latter was only begun in 1950; moreover, the first works almost totally ignored the achievements of one-dimensional hydrodynamics [2] and gas dynamics [3, 4] connected with the calculation of flows in pipes. This is explained by various factors, including the characteristics of the operation of the wells and the variety of goals which are set during the investigation of heat exchange in wells and pipelines. The first factor will be considered below; as for the second factor, its influence is clear, if only because of the fact that the first investigations of the thermal regime of wells were due to the requirements of geothermal studies and geophysical exploration [5, 6].

In a series of subsequent works [7, 8, 9], questions of heat exchange between wells and rock were studied in connection with an investigation of the effect of the circulation of flushing fluid during boring on the heat field of the ground around the well.

Work [7] deserves special mention, since this work for the

first time precisely formulated the quasistationary problem of heat exchange between a flow of liquid in a pipe and rock. Subsequently this approach to the investigation of the thermal regime of wells was used rather widely; in the domestic literature its author is considered to be I. A. Charnyy, who independently carried out an analogous investigation [9] several years after work [7].

The basic idea of works [7, 9] consisted in the assumption regarding the significant difference in the time scales (if one uses the contemporary terminology of [10]) of heat exchange in pipes (forced convection) and in rock (thermal conductivity). Under this physically obvious assumption, the substationary derivative in the energy equation may be neglected, and the coefficient of heat transfer from the liquid to the rock is considered to depend weakly on time. The difference between works [7] and [9] consists only in the choice of the form of the dependence of this coefficient on time; in the first work this dependence is determined from an exact solution of the corresponding thermal conductivity problem, and in the second work, from an approximate solution, obtained using the method of quasistationary states. Subsequent comparison with the exact solution showed that the results of work [9], generally speaking, are suitable for sufficiently large values of time [11].

The ideas of work [7] are found also in the investigation of H. Ramey [12], who examined the thermal regime of oil and gas wells in the quasistationary approximation. Considering the gas to be ideal, and the oil to be an incompressible fluid, he obtained from

the equation of energy conservation an ordinary differential equation of the first order for determining the temperature. The dependence of the heat transfer coefficient on time was determined from an approximate solution of an axisymmetric problem for a heat conduction equation suitable for large values of time.

As was indicated by V. N. Petrov [13], the formulas obtained by H. Ramey give sufficiently accurate results when calculating the temperature in water and oil wells for the case where they operate for a sufficiently long period of time. As for gas wells, by virtue of the assumptions used (the ideal gas assumption) Ramey's formulas are valid for wells that are not deep (on the order of 1000m) and for pressure gradients that are not large.

An attempt to take into account the real properties of a gas was undertaken for the first time by S. A. Bobrovskiy and V. I. Chernikin [14], who investigated the stationary nonisothermal movement of gas in a well taking into account the Joule-Thomson effect for the case of a constant mass flow. The temperature of the rock was assumed to be linearly dependent on the depth, and the coefficient of external heat exchange and the orificing coefficient were assumed to be constant. In addition, it was assumed that the pressure was a linear function of the depth of the well.

Later U. P. Korotayev analyzed in detail the stationary nonisothermal flow of a real gas in a well for the case of a rock temperature which depended linearly on depth. The results of his investigations [15] amount essentially to the following.

For the equation of state of a gas that was proposed by K. V. Pokrovskiy, integration of the system of ordinary differential equations describing the motion of a real gas in a well may be carried out using the method of successive approximations, where the zeroeth approximation corresponds to the case of an ideal gas. The general solution is constructed in the form of a series in powers of a small parameter. By means of comparison with the exact numerical solution it is shown that for the majority of practically interesting cases the first approximation is sufficient. However, even in this case the solution is obtained in the form of cumbersome quadratures, which compelled the author to look for the possibility of further simplification. Unfortunately, the method selected by him is not always convincing. For example, it is pointed out that a taking into account of the real properties of the gas has a greater effect on the character of the temperature curve than on the pressure gradient between the bottom and mouth of the well. However, following this the temperature is averaged over the depth of the well in order to simplify the method of calculation of the pressure.

It is necessary to note that the results obtained by U. P. Korotayev may be easily extended to the case where one takes into account the quasistationary heat exchange between the gas and the rock.

Further investigations of the thermal regime of operating wells are connected with the solution of nonstationary problems of heat

exchange between a flow of liquid or gas and rock. Problems of this sort belong to the class of conjugate problems of heat exchange [10].

In connection with the study of the thermal regime of operating wells, the first problem of this sort was examined in 1957 by L. Lesem and coauthors [16], who, for the case of an ideal gas and with the aid of the Laplace transform, solved a system of linear differential equations in partial derivatives that consisted of the energy equation for the gas in the well and the heat conduction equation in rock. In the latter equation, heat conduction along the borehole was neglected. The conditions of conjugation between the heat flows in the well and in the rock corresponded to the case of the absence of heat-transfer resistance. The solution was obtained in the form of complicated integrals of Bessel functions, the graphs of which are given in [16].

The results of L. Lesem et al. were used in work [17], which was devoted to a theoretical investigation of the thermal regime of water-injection wells. Comparison of the results of the calculations with the data of the conducted experiments proved to be satisfactory. The authors of work [17] note the difficulties in calculating the quadratures obtained by L. Lesem et al. in [16], and derive simple asymptotic formulas which are suitable for large values of time. Later, N. A. Avdonin and A. A. Buykis obtained a formula for small values of time and showed that it corresponds to the solution of the problem of the injection of fluid through

a gallery [18]. Earlier this same result was obtained by E. B. Chekalyuk using a somewhat different approach [19].

With the advent of high-speed computers there arose the possibility of a qualitatively new approach to the investigation of the thermal regime of wells. This possibility was successfully realized by V. N. Petrov in his doctoral dissertation, completed in 1968. Unfortunately, the fundamental results of this large and interesting work were published in a variety of low-circulation collections, for example [20], and this work turned out to be less well-known than it deserves to be.

The work of V. N. Petrov is primarily concerned with an analysis of the various effects caused by taking into account the real properties of a gas and by the effect of these properties on the thermophysical characteristics of the gas. The calculations carried out by him showed that the differences in the mouth temperature values for calculation variants distinguished from each other by the equation of state and by the nature of the dependence of the heat capacity on pressure and temperature attain 10-15°C and increase with time. All the calculations were made for a constant mass flow of the gas and under several physically well-founded assumptions. The work contains a series of conclusions that are important from a practical standpoint; of these, we note the results relating to the integration of the gradient-thermogram of gas wells. V. N. Petrov showed that in exploited wells the gradient-thermograms are

relatively stable, and the amplitudes of anomalies attain a maximum in the first few hours after the well is started up.

In recent years the thermal regime of gas wells has attracted the attention of investigators in connection with the problem of preventing hydrate formation. This problem is of special interest for regions where long-term frost occurs, since here the problem of determining the thermal regime of wells becomes especially difficult due to the presence of a moving interface between thawed and frozen rock. The determination of the equation of motion of this boundary and the temperature fields in the thawed and frozen zones in itself presents considerable mathematical difficulties [21], which are compounded upon the simultaneous examination of the heat fields in the rock and the heat flow in the well. It is not surprising that the number of works devoted to this subject is very small.

First, let us mention the investigations in which the effect of long-term frost is taken into account by means of partitioning the region of integration into two zones with different geothermal curves, corresponding to ordinary and long-frozen rock [15, 22, 23]. It is obvious that such an approach cannot give satisfactory results, since here the role of the ice-to-water phase transition is not taken into account.

B. L. Krivoshein and A. A. Koshelev [24] have proposed an algorithm for solving the problem of the determination of the temperature of operating wells in a region of long-term frost and have

announced the possibility of its realization on a computer, where it is implicitly assumed that it is possible to separate the problem of the motion of the gas from Stefan's problem. Works [25, 26] formulate and solve for the first time the problem of the heat exchange between an operating well and long-frozen rock. The content of these works will be presented in the appropriate section of this book.

Section 2. Methods for Solving Problems of Heat Exchange Between Wells and Frozen Rock

The intensity of the heat transfer from the flow in a well into the ground is determined by the nature of the temperature field of the ground in the region of the borehole. During heat exchange between the flow (which has, as a rule, a positive temperature) and the surrounding frozen ground, there occur in the latter phase transitions accompanied by the absorption of a significant amount of heat. This leads to the fact that the rate of movement of the zero isotherm in such ground sharply decreases in comparison with ground in which there are no phase transitions, and the intensity of heat transfer for the corresponding moments in time increases. Thus, in order to calculate the temperature of a flow moving in a well, it is necessary to obtain a solution of the problem of the thawing of frozen rock in the space around the well. Very many works are devoted to solving this problem. In this paragraph we shall consider only the most significant of these works.

At the present time, the majority of geocryologists accept

the following definition of frozen rock: "By frozen rock, ground, and soil is meant rock, ground, and soil which possess a negative or zero temperature and in which at least part of the water has made the transition into the crystalline state [27]."

Upon the supply of a sufficient amount of heat to the frozen rock, the ice cementing the separate particles of rock fully or partially turns into water. This sharply worsens the cohesiveness of the rock, which then serves as a reason for the formation of subterranean cavities, griffons, gas leakage in the space around the well, and other complications.

The water in the frozen rock interacts with the active surfaces of the mineral skeleton. This lowers the temperature at which the water freezes. Simplifying somewhat the rather complex physical picture of this phenomenon, one may say that the thinner the capillary tube in which the water is located, the lower its freezing temperature. Thus, the water in the frozen ground freezes in some spectrum of temperatures. Moreover, it may be confirmed that at any negative temperature some amount of the water in the frozen ground will be in the unfrozen state.

On this basis, A. G. Kolesnikov [28] proposed the following formulation of the problem of the freezing of the ground. The iciness of the ground i is assumed to be a known function of the temperature Θ, i.e. $i = i(\Theta)$. The thermophysical properties of the ground depend on the iciness: $c = c(i)$; $\lambda = \lambda(i)$. A change in the iciness Δi leads to the liberation (absorption) of

the latent heat of fusion $\rho l w \Delta i$, where w is the moisture content.

Thus, the heat conduction equation for freezing moist ground (for the one-dimensional case) assumes the form

$$c(i)\rho\frac{\partial\Theta}{\partial t}=\frac{\partial}{\partial x}\left[\lambda(i)\frac{\partial\Theta}{\partial x}\right]-\rho l w\frac{\partial i(\Theta)}{\partial t}. \qquad (1.1)$$

In this formulation, the heat of fusion is taken into account in the form of internal sources of heat that are distributed over volume; the power of these sources depends on the temperature. S. H. Cho and J. E. Sunderland [29] analogously examine the problem of freezing or thawing. The complexity of the obtained equations was the reason that the solutions of the problems of the freezing or thawing of the ground in this formulation have been obtained only for the simplest cases [30], [31]. Another solution of the problem of freezing or thawing, known as Stefan's problem, has found much wider application.

Study of the iciness curves of frozen ground shows that the overwhelming part of the water in the ground freezes in a spectrum of negative temperatures, localized around 0°C. This spectrum is very narrow for coarse rock and is somewhat wider for fine rock. Proceeding on the basis of this fact, the assumption is made that all the phase transitions of the water in the ground proceed at 0°C and that the latent heat of fusion is liberated only at the interface between the solid and liquid phases. In this formulation the problem

of the thawing of frozen ground reduces to the solution of the following equations (for the one-dimensional case):

$$\frac{\partial \bar{\Theta}_1}{\partial \bar{t}} = a_1 \frac{\partial^2 \bar{\Theta}_1}{\partial x^2}; \quad 0 < x < \bar{S}(\bar{t}) \tag{1.2}$$

$$\frac{\partial \bar{\Theta}_2}{\partial \bar{t}} = a_2 \frac{\partial^2 \bar{\Theta}_2}{\partial x^2}; \quad \bar{S}(\bar{t}) < x < b \leq \infty \tag{1.3}$$

The boundary conditions are as follows:

$$\bar{\Theta}_1 = T > 0 \quad \text{for} \quad x = 0; \tag{1.4}$$

$$\bar{\Theta}_2 = \bar{\Theta}_0 \leq 0 \quad \text{for} \quad t = 0; \tag{1.5}$$

$$\bar{\Theta}_1 = \bar{\Theta}_2 = 0 \quad \text{for} \quad x = \bar{S}; \tag{1.6}$$

$$-\bar{\lambda}_1 \frac{\partial \bar{\Theta}_1}{\partial x} + \bar{\lambda}_2 \frac{\partial \bar{\Theta}_2}{\partial x} = \rho l w \frac{d\bar{S}}{d\bar{t}} \quad \text{for} \quad x = \bar{S}. \tag{1.7}$$

The problem of the freezing of moist ground is posed analogously.

The first solution of Stefan's problem was found in 1831, when G. Lame and B. P. Clapeiron [32] found a similitude solution for the case $\Theta_F = 0$. A similitude solution for $\bar{\Theta}_c \neq 0$ was given by F. Neumann [33], J. Stefan [34], and C. Schwartz [35]. N. N. Verigin [36] found a similitude solution for the axisymmetric Stefan's problem for a linear cold source (for the case where the ground freezes). Exact solutions of separate problems were obtained in the works of I. G. Portnov [37] and G. A. Tirskiy [38]. These works practically exhaust the number of exact solutions of problems of the Stefan problem type. In the overwhelming majority of cases (nonuniform problems, problems with cylindrical symmetry, problems with various boundary and initial conditions, etc.), numerical, analog, and approximation methods are used to solve these problems. Let us mention the most important of these methods.

In the works of L. I. Rubinshteyn [21, 39], A. Friedman [40], J. J. Kolodner [41], and other authors, there was developed a method for reducing Stefan's problem to a system of integral equations of the Volterra type. Solution of the latter is carried out with the aid of iterations on a computer.

V. G. Melamed [42] developed a method for reducing Stefan's problem to a system of an infinite number of ordinary differential equations. Truncating this system to n equations gives an approximate solution, which as $n \rightarrow \infty$ approaches the exact solution. An advantage of the method is the possibility of using standard programs

for integrating the obtained system on a computer.

A large number of works are devoted to the development of finite-difference methods for solving Stefan's problem. As an example, we may mention the works of I. Douglas [43], B. M. Budak [44], and G. M. Dussinberre [45, 46].

Analog methods with the use of hydrointegrators and electronic differential analyzers have found wide application in the solution of Stefan's problem. I. M. Kutasov [47], G. I. Man'kovskiy [48], F. Ya. Novikov [49], and others investigated the temperature fields around freezing and heating wells using hydrointegrators. Hydrointegrators of the system of V. S. Luk'yanov are often used.

A study of the formation of ice with the aid of analog computers was made by R. Howe [50] and A. L. London [51]. A problem taking into account variable thermophysical properties was investigated on an electronic differential analyzer by D. R. Otis [52].

In view of the fact that an exact solution of Stefan's problem is connected with great computational difficulties, various approximation methods have found widespread application. Let us examine the most important of these methods.

1. The method of the successive replacement of stationary states. The idea of the method belongs to L. S. Leybenzon, who used the method in solving a problem concerning the hardening of oil in a pipe [53]. He assumed that because of the small rate of advancement of the hardening front, the temperature distribution in the solid phase at each moment in time differs little from the

stationary distribution for the given position of the hardening front. The temperature of the liquid phase was assumed to be equal to the pour point.

Thus, the temperature and heat flow at each point of the solid phase were determined from a solution of the stationary problem as a function of the coordinate of the point and the coordinate of the boundary. The obtained expression for $\left.\frac{\partial \bar{\Theta}}{\partial x}\right|_{x=\bar{S}}$ was substituted into Stefan's condition, which led to an ordinary differential equation for $\bar{S}(\bar{t})$. This method, because of its simplicity, has found wide application in engineering calculations [54].

One may consider the work of F. Kreith and F. E. Romie [55] to be an evolvement of this method. The essence of this work reduces to the following. Actually, in the method of L. S. Leybenzon the temperature distribution is determined from equation (1.2) under the assumption that the inertia term $\frac{\partial \bar{\Theta}_1}{\partial \bar{t}}$ is equal to zero. In [55] this solution is taken as the zeroeth approximation $\bar{\Theta}_1^{(0)}$. One looks for a solution to the problem having the form

$$\bar{\Theta}_1 = \sum_{n=0}^{\infty} \bar{\Theta}_1^{(n)},$$

where $\bar{\Theta}_1^{(n)}$ is determined from the equation

$$a_1 \frac{\partial^2 \bar{\Theta}_1^{(n)}}{\partial x^2} = \frac{\partial \bar{\Theta}_1^{(n-1)}}{\partial \bar{t}}.$$

The problems of the hardening of fluid in a pipe and in a sphere for a zero initial temperature are solved in [55] using this method. The method is analogous to the one which was used by M. Ye. Shvets [56] and Ye. M. Dobryshman [57] in the solution of the equations of a boundary layer.

2. The integral method in its contemporary formulation was developed by T. Goodman [58] and I. Green [59], but it was used earlier by I. A. Charnyy [60] in solving plane and axisymmetric Stefan problems.

The essence of the method consists in replacing the heat flow equation by the heat balance relation. The latter is obtained by equating the increment in the heat accumulated by the body per unit time to the heat flow through the moving boundary. T. Goodman obtains the relation formally, by integrating the heat flow equation over the spatial coordinate within the limits of a single phase, and I. A. Charnyy obtains it from physical considerations.

Next, the temperature profiles are presented in the form of polynomials with coefficients depending on the coordinate of the boundary (for the linear problem), or in the form of a logarithmic profile (for the axisymmetric problem). In order to obtain such a profile in a semi-infinite region, one introduces the concept of the zone of thermal influence, at the boundary of which the heat flow is assumed to be equal to zero, and the temperature, equal to the unperturbed temperature of the body. The obtained temperature profiles are substituted into the heat balance relation and into

Stefan's condition, which leads to a system of ordinary differential equations with respect to the unknown functions.

Kh. R. Khakimov [61], using this method to solve the problem of the freezing of the ground around a freezing borehole, assumed, in order to simplify the problem, that the radius of the front of thermal influence was proportional to the radius of the thawing front. The coefficient of proportionality was determined by him experimentally. V. A. Maksimov et al. [62] used the integral method in solving the problem of the melting of a semi-infinite rod in a medium having a constant or varying temperature, with ablation or without it. In the aforementioned work [29] the integral method was used in solving a problem with an extended phase transition front. T. Goodman [63] used this method in solving Stefan's problem for a body with variable thermophysical properties.

3. The method of continuations. A significant feature of this method is the fact that a fictitious body is introduced, the shape of which is invariant with time and coincides with the shape of the body in question up to the beginning of the phase transformations. Thus, in connection with problem (1.2)-(1.7), the region in question is $0 \leq x \leq d \leq \infty$, in which equation (1.3) is satisfied. In order that the conditions be satisfied at the moving boundary, a fictitious heat flow is introduced at the fixed boundary. After satisfying the boundary conditions, one arrives at an integrodifferential equation for determining the heating capacity of the fictitious heat flow. This equation is solved numerically or by expanding the

desired solution in a series. Such a method was developed by B. A. Boley, who used this method in solving one-dimensional [64] and axisymmetric [65] Stefan problems with ablation and without it.

V. I. Antipov and V. V. Lebedev [66], having solved problem (1.2)-(1.7) with variable initial and boundary conditions, introduced two fictitious temperatures: one, depending on time, into condition (1.4); the other, depending on the coordinate, into condition (1.5). After satisfying the conditions at the moving boundary, they arrived at a system of two integrodifferential equations, which were solved by expanding the unknown functions in a series.

G. A. Martynov [67] used this method in solving the inverse Stefan problem.

4. The method of contour integrals. The method was proposed by G. A. Grinberg [68] for solving problem (1.2)-(1.7) for $\overline{\Theta}_F = 0$. The new variable $z = x - \overline{S}(\overline{t})$ is introduced. Then equation (1.2) acquires the form

$$\frac{\partial^2\overline{\Theta}_1}{\partial z^2} + \frac{d\overline{S}}{d\overline{t}} \cdot \frac{\partial\overline{\Theta}_1}{\partial z} = \frac{\partial\overline{\Theta}_1}{\partial\overline{t}}. \tag{1.8}$$

The function

$$\exp[\lambda\overline{t} - \sqrt{\lambda}(z + \overline{S}(\overline{t}))],$$

where λ is any complex constant, is a particular solution of equation (1.8). The general solution may be given in the

form

$$\overline{\Theta}_1 = \frac{1}{2\pi i} \int_{(\Gamma)} c(\lambda) \exp\left[\lambda \bar{t} - \sqrt{\bar{\lambda}}\,(z + \bar{S}(\bar{t}))\right] d\lambda, \tag{1.9}$$

where c(λ) is an arbitrary function. The contour of integration Γ is chosen in the form of a line parallel to the imaginary axis in such a way that all the singular points of the integrand remain to the left of the line. From condition (1.6) we obtain

$$\frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} c(\lambda) \exp\left[\lambda \bar{t} - \sqrt{\bar{\lambda}}\,\bar{S}(\bar{t})\right] d\lambda = 0. \tag{1.10}$$

From this relation the form of the function c(λ) is determined.

The method is used advantageously when solving the inverse Stefan problem, i.e. when determining the temperature fields from a given equation describing the movements of the boundary [67, 69], but this method may also be used for solving the direct problem under the conditions of a constant initial temperature and a constant temperature at the boundary.

5. The variational method (Biot's method). M. A. Biot is justly considered to be the author of this method, in spite of the fact that the variational formulation of the theory of heat conduction was first proposed by Rosen [70]. The point is that M. A. Biot not only developed the technique for applying the variational principle to dissipative processes, but also derived general differential equations of the Lagrange type for a thermodynamic system subjected

to irreversible changes [71]. Later, L. G. Chambers used this technique for solving heat conduction problems [72].

The essence of the method of M. A. Biot consists in the following [73]. One introduces the heat flow vector field

$$Q = Q(s_i, x_i),$$

where Q is the amount of heat per unit area that has passed through the given cross section in a given direction since the moment the process began; s_i are the generalized coordinates; and x_i are the spatial coordinates. The following invariants are introduced: the thermal potential

$$u = \frac{1}{2} \int_v c\rho\Theta^2 dv$$

and the dissipative function

$$D = \frac{1}{2} \int_v \frac{q^2}{\lambda} dv.$$

Here v is the volume of the body in question; θ is the temperature; $q = \frac{\partial Q}{\partial t}$; and λ is the coefficient of thermal conductivity. Then the variational principle is formulated in the following manner:

$$\delta u + \delta D = \int_F \Theta \delta Q dF. \qquad (1.11)$$

where F is the surface of the body. This relation is equivalent to the heat conduction equation.

If one introduces the concept of the thermal force w_i

$$\sum w_i \delta s_i = \int_F \Theta \delta \dot{Q} dF,$$

then the variational principle may be written in the form of Lagrange's equation:

$$\frac{\partial u}{\partial s_i} + \frac{\partial D}{\partial \dot{s}_i} = w_i. \qquad (1.12)$$

In order to apply this principle to Stefan's problem, the latent heat of fusion is introduced in place of Q, and the coordinate of the moving boundary is used as the generalized coordinate. Given a temperature profile in the form of a polynomial, one arrives at an ordinary differential equation for $\overline{S}(\overline{t})$.

As is apparent from what has been presented, there exist a vast number of methods for solving Stefan's problem; these methods have been developed mainly for linear and axisymmetric cases. For the solution of engineering problems concerning the thawing of frozen ground around a well in a plane formulation, the method of successive approximations is most acceptable. This method combines, as will be shown below, simplicity and a high accuracy of the obtained solutions. At the same time, practically no approximation methods exist for solving two- and three-dimensional Stefan problems, which

arise, in particular, when calculating the thermal regimes of wells bored and exploited in a region of long-term frost.

CHAPTER 2
HEAT EXCHANGE BETWEEN FLOWS IN A WELL AND ROCK

Section 1. Derivation of a System of Equations for the Nonisothermal Motion of Liquid and Gas in Pipes

In accordance with the fundamental assumptions of continuum mechanics, for the description of the nonisothermal flow of a compressible fluid in a well we use three conservation laws: conservation of mass, momentum, and energy.

1. The law of conservation of mass or the equation of continuity in the case of the presence of internal sources of mass has the form [74]

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \vec{V}) = \dot{m}, \tag{2.1}$$

where ρ is the density; $\overline{V}$ is the velocity vector; $\dot{m}$ is the rate of increase in the mass per unit volume due to internal sources. In what follows we shall everywhere, with the exception of special cases, consider that $\dot{m} = 0$.

In the case of one-dimensional motion in pipes having constant cross sections, from (2.1) we obtain

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho v) = 0. \tag{2.2}$$

In practice wells most often operate with a constant output, which corresponds to

$$\rho v f = M = \text{const} \tag{2.3}$$

and

$$\rho = \rho(x). \tag{2.4}$$

Here f is the cross section of the pipe, and M is the mass flow. For an incompressible fluid (ρ = const), from (2.2) it follows that

$$\frac{\partial v}{\partial x} = 0, \tag{2.5}$$

or

$$v = v(t). \tag{2.6}$$

2. The dynamic equation. In order to derive the equation the following theorem is used: the change in the main vector of the momenta of a system of material points is equal to the impulse of all the external mass and surface forces [74].

The application of this theorem in pipe hydraulics gives [75]

$$\frac{\partial (\rho v)}{\partial t} + \frac{\partial}{\partial x}(p + \rho v^2) = -\rho g \sin\alpha - \frac{\lambda}{2D}\rho |v| v, \tag{2.7}$$

where p is the pressure; g is the acceleration due to gravity; α is the angle of inclination of the pipe, measured from the horizontal; λ is the coefficient of hydraulic resistance; and D is the diameter of the pipe.

For wells one may take $\sin \alpha = 1$. If one uses (2.3), then from (2.7) we obtain

$$\frac{\partial}{\partial x}(p + \rho v^2) = -\rho g - \frac{\lambda}{2D}\rho |v| v. \tag{2.8}$$

Further simplification of the equation of motion is connected with the possibility of neglecting the velocity head, since the rates of motion of gas during normal operation of wells are comparatively small [76]. In the case of accidental gushing this assumption will no longer be justified. The simplified equation of motion has the form

$$-\frac{\partial p}{\partial x} = \gamma\left(1 + \frac{\lambda}{2gD}|v|\, v\right). \tag{2.9}$$

where $\gamma = \rho g$ is the specific gravity of the gas.

3. The law of conservation of energy. As the initial differential equation let us use the equation derived in work [75]:

$$\frac{\partial}{\partial t}\left[\rho\left(g\bar{z} + Iu + \frac{v^2}{2}\right)\right] + \frac{\partial}{\partial x}\left[\rho v\left(g\bar{z} + Iu + \frac{p}{\rho} + \frac{v^2}{2}\right)\right] = = \frac{4}{D} Iq + I\frac{\partial}{\partial x}\left(k\frac{\partial \bar{T}}{\partial x}\right). \tag{2.10}$$

Here $\bar{z}(x)$ is the ordinate of the axis of the pipe, measured from the horizontal plane; I is the mechanical equivalent of heat; u is the unit internal energy; q is the heat flow through the wall of the pipe; and k is the coefficient of thermal conductivity of the gas.

Let us transform the left hand side of equation (2.10) in the following manner:

$$\frac{\partial}{\partial \bar{t}}\left[\rho\,(g\bar{z}+E)\right]+\frac{\partial}{\partial x}\left[\rho v\left(g\bar{z}+\frac{p}{\rho}+E\right)\right]=\rho\frac{\partial}{\partial \bar{t}}(g\bar{z}+E)+$$
$$+(g\bar{z}+E)\frac{\partial\rho}{\partial \bar{t}}+\rho v\frac{\partial}{\partial x}\left(g\bar{z}+E+\frac{p}{\rho}\right)+\left(g\bar{z}+E+\frac{p}{\rho}\right)\frac{\partial(\rho v)}{\partial x}=$$
$$=\rho\frac{d}{d\bar{t}}(g\bar{z}+E)+\rho v\frac{\partial}{\partial x}\left(\frac{p}{\rho}\right)+\frac{p}{\rho}\cdot\frac{\partial(\rho v)}{x\partial},$$
$$\left(E=Iu+\frac{v^2}{2}\right).$$

Now equation (2.10) may be put into canonical form [74]:

$$\rho\frac{dE}{d\bar{t}}=-\rho g\bar{z}'v-\frac{\partial(pv)}{\partial x}+\rho Q, \qquad (2.11)$$

where

$$Q=\frac{4}{D\rho}Iq+\frac{I}{\rho}\cdot\frac{\partial}{\partial x}\left(k\frac{\partial \bar{T}}{\partial x}\right).$$

In the majority of practically interesting cases the heat conduction may be neglected, and then the equation of energy during motion in the well acquires the form [next page]:

$$\rho \frac{dE}{dt} = -\rho g v - \frac{\partial(p v)}{\partial x} + \frac{4}{D} I q. \qquad (2.12)$$

In the general case the system of equations (2.2), (2.8), and (2.12), augmented by the equation of state

$$p = \rho g Z R \bar{T}, \qquad (2.13)$$

is a system of equations of the hyperbolic type, possessing three real characteristics [75]:

$$\frac{dx}{dt} = v, \quad \frac{dx'}{dt} = v \pm c, \qquad (2.14)$$

where

$$c = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_s}$$

is the speed of sound in the gas.

If the external heat flow is a given function of x and t, then, as is shown in [75], system (2.2), (2.8), (2.12), and (2.13) may be reduced either to characteristic form and integrated by the method of characteristics, or to a system of finite-difference equations based on an implicit absolutely stable difference scheme [77].

The matter is more complicated in the case where the external

heat flow depends on the nature of the heat exchange between the flow in the well and the rock. Then to the given system of equations it is necessary to add an equation describing the propagation of heat in the rock, and conditions of conjugation of the heat flows. For rock having constant thermophysical properties this equation will be Fourier's equation

$$\frac{\partial \bar{\Theta}}{\partial \bar{t}} = a_1 \nabla^2 \bar{\Theta}.$$

It is natural to assume that the heat from the well propagates mainly in the radial direction, which allows one to write the last equation in the form

$$\frac{\partial \bar{\Theta}}{\partial \bar{t}} = a_1 \left(\frac{\partial^2 \bar{\Theta}}{\partial \bar{r}^2} + \frac{1}{r} \cdot \frac{\partial \bar{\Theta}}{\partial \bar{r}} \right), \tag{2.15}$$

where $\bar{\Theta}$ is the temperature of the rock, and a_1 is the coefficient of thermal diffusivity. At the contact between the rock and the well one must satisfy the boundary condition

$$\bar{\lambda}_1 \frac{\partial \bar{\Theta}}{\partial \bar{r}} = \bar{\alpha}(\bar{\Theta} - \bar{T}) \quad \text{for} \quad \bar{r} = a, \tag{2.16}$$

where $\bar{\lambda}_1$ is the thermal conductivity of the rock; $\bar{\alpha}$ is the total heat transfer coefficient; and a is the radius of the borehole. In what follows it is convenient to operate with the temperature of the gas; therefore, let us transform the energy equation (2.12).

Reading off from (2.12) the equation

$$\rho \frac{d}{d\bar{t}}\left(\frac{v^2}{2}\right) = -\rho g v - \frac{\partial(\rho v)}{\partial x} + N_{in}, \tag{2.17}$$

expressing the theorem regarding the change in the kinetic energy in differential form [74], we obtain

$$\rho \frac{du}{d\bar{t}} = \frac{4}{D} lq - N_{in}, \tag{2.18}$$

where N_{in} is the power of the internal forces per unit volume.

For the case of one-dimensional motion of the gas,

$$N_{in} = Ap\frac{\partial v}{\partial x},$$

which, taking into account the equation of continuity, may be put in the form

$$N_{in} = -\frac{Ap}{\rho}\frac{d\rho}{d\bar{t}}. \tag{2.19}$$

If we now pass from the internal energy to the enthalpy

$$i = u + \frac{Ap}{\rho} \tag{2.20}$$

and express q in terms of the heat flow at the wall of the borehole

$$q = \bar{\lambda}_1 \frac{\partial\bar{\theta}}{\partial\bar{r}}\bigg|_{\bar{r}=a}, \tag{2.21}$$

then equation (2.18) acquires the form

$$\rho \frac{di}{d\bar{t}} = A \frac{dp}{d\bar{t}} + \frac{2}{a} \bar{\lambda}_1 \left. \frac{\partial \bar{\Theta}}{\partial \bar{r}} \right|_{\bar{r}=a} \tag{2.22}$$

Using the expression for the enthalpy of a gas

$$di = c_p (d\bar{T} - \mu_i dp), \tag{2.23}$$

where μ_i is the Joule-Thomson coefficient, we obtain

$$\rho c_p \frac{d\bar{T}}{d\bar{t}} - (\rho c_p \mu_i + A) \frac{dp}{d\bar{t}} = \frac{2}{a} \bar{\lambda}_1 \left. \frac{\partial \bar{\Theta}}{\partial \bar{r}} \right|_{\bar{r}=a}$$

or, passing to partial derivatives and assuming that the velocity and pressure of the gas do not depend on time,

$$\rho c_p \frac{\partial \bar{T}}{\partial \bar{t}} + \rho c_p v \frac{\partial \bar{T}}{\partial x} - v (\rho c_p \mu_i + A) \frac{\partial p}{\partial x} = \frac{2}{a} \bar{\lambda}_1 \left. \frac{\partial \bar{\Theta}}{\partial \bar{r}} \right|_{\bar{r}=a} . \tag{2.24}$$

The latter assumption is based on the fact that the redistribution of the pressure proceeds much more rapidly than the redistribution of the temperature, and consequently, if the well operates with a constant output, then a very short time after the well is started up it enters into the quasistationary regime, in which the slow changes in the pressure and velocity over time will be due to heat exchange with the rock and to the effect of orificing [20].

If now we introduce the mass flow of the gas in accordance with

(2.3), then in place of (2.24) we obtain

$$\rho f c_p \frac{\partial \bar{T}}{\partial \bar{t}} + c_p M \frac{\partial \bar{T}}{\partial x} - \left(\mu_i + \frac{A}{\rho c_p}\right) c_p M \frac{\partial p}{\partial x} = 2\pi a \bar{\lambda}_1 \frac{\partial \bar{\Theta}}{\partial \bar{r}}\bigg|_{\bar{r}=a}. \qquad (2.25)$$

For the case of a constant mass flow, the system of equations (2.9), (2.13), and (2.25) must be augmented by two boundary conditions and one initial condition. Usually, the pressure and temperature at the bottom are given as boundary conditions. At the same time, it is clear that these values must be determined from the solution of the corresponding problem of the nonisothermal filtration of the gas or using the well-known relations of subterranean hydrodynamics, if one is talking about a liquid (oil).

Section 2. Determination of the Temperature of Gas at the Bottom of Wells

In order to obtain a complete description of the processes which occur during the filtration of a liquid or a gas in a porous medium, it is necessary, analogous to what was done above, to derive a system of equations of continuity, motion, and energy. Such a derivation is carried out in a series of works [19, 78-80]. However, the obtained system of equations is so complex that the solution of the corresponding boundary problems turns out to be possible only after a series of simplifications or with the use of numerical methods. This is not surprising, if one recalls that the solution of the problems of isothermal filtration in itself presents considerable difficulties.

For these reasons, analysis of the temperature fields during the filtration of a liquid or gas is made mainly under the assumption that the effect of the temperature on the nature of the redistribution of the pressure is negligibly small [19, 78]. With such an approach it is considered that the pressure field is known from the solution of the corresponding problem of subterranean hydrodynamics, and once it is so, substitution of the necessary relations into the energy equation allows one to find in principle the temperature of the gas or liquid.

The assumption mentioned above is based on the fact that the rate of change of the pressure is much greater than the rate of change of the temperature. This actually takes place during the filtration of an elastic liquid in strata with good collector properties, but is not necessarily true during the filtration of gas in deeply located strata with low permeabilities. For example, for stationary filtration of an ideal gas, from Darcy's law and from the system of the equations of continuity

$$\operatorname{div}(\gamma \vec{w}) = 0 \tag{2.26}$$

and energy

$$\bar{\lambda}_1 \operatorname{div} \operatorname{grad} \bar{T} - c_p \gamma \vec{w} \operatorname{grad} \bar{T} = 0 \tag{2.27}$$

it follows that this process is described by Laplace's equation for

the function

$$u = p^2 + 2bT, \tag{2.28}$$

where p, T are the dimensionless pressure and temperature respectively;

$$b = \frac{\bar{\lambda}_1}{c_p \gamma_0} \cdot \frac{p_0}{p_i} \cdot \frac{T_i}{T_0} \cdot \frac{1}{m\varkappa}; \tag{2.29}$$

γ_0 is the bulk density of the gas at pressure p_0 and temperature T_0; m is the porosity; $\varkappa$ is the coefficient of piezoconductivity for an initial stratum pressure of p_i; and T_i is the initial stratum temperature.

By analogy with the problems of thermoelasticity, let us call b the coupling parameter. In the given case it characterizes the increase in pressure due to coupling of the velocity and temperature fields. Estimates show that under the usual conditions this parameter is on the order of 10^{-2} to 10^{-3}. However, for strata with good thermophysical and poor collector properties, it may have a considerably higher value.

Thus, for the case of a small coupling parameter, for determining the temperature at the bottom of boreholes one may use the approximate formula of E. B. Chekalyuk [19]:

$C_\pi \equiv C_r$
$r_c \equiv r_b$

$$\Delta T = \mu_i \frac{\Delta p}{\ln \frac{R_k}{r_c}} \ln \sqrt{1 + \frac{c_p G t}{\pi h c_n r_c^2}}. \tag{2.30}$$

where

$$R_k = R_k(\bar{t})$$

is the radius of the zone of influence of the borehole; r_b is the radius of the borehole; h is the power of the stratum; c_r is the volumetric heat capacity of the rock; G is the mass yield of the well; and $\bar{t}$ is the time.

If the dependence

$$R_k = R_k(\bar{t})$$

is determined using the well-known formula of E. B. Chekalyuk [19], then after straightforward algebra expression (2.30) acquires the form

$r_c \equiv r_b$
$c_\pi \equiv c_r$

$$\Delta T = \frac{\varepsilon_i \Delta p}{\ln\left(1 + \sqrt{\frac{\pi \varkappa \bar{t}}{r_c^2}}\right)} \ln \sqrt{1 + \frac{c_p G \bar{t}}{\pi h c_\pi r_c^2}}. \qquad (2.31)$$

Calculations using formulas (2.30)-(2.31) may be made for the case of small depressions. If the depressions are great, then this invalidates the assumptions regarding the constancy of the thermodynamic characteristics of the system that were made in deriving these formulas.

In this case, let the coupling parameter again be small. Then

the pressure may be determined using the well-known formulas of subterranean hydraulics obtained for isothermal filtration, for example using the formula [79]

$$p^2 = p_k^2 - \frac{\mu G \rho_0}{\pi k h \gamma_0} \ln \frac{R_k}{r}, \qquad (2.32)$$

where μ is the viscosity of the gas, and k is the permeability, and then, using the energy equation for the case of negligibly small heat conducting flows

$$\frac{d\overline{T}}{d\overline{r}} + \mu_i \frac{dp}{d\overline{r}} = 0, \qquad (2.33)$$

one may determine the temperature from the solution of the ordinary differential equation of the first order

$$\frac{d\overline{T}}{d\overline{r}} + \mu_i [\overline{T}, p(\overline{r})] f(\overline{r}) = 0, \qquad (2.34)$$

obtained by substituting relation (2.32) into equation (2.33).

If the coupling parameter cannot be considered to be negligibly small, then it is necessary to determine the temperature from the solution of a system of ordinary nonlinear differential equations, containing besides (2.33) the equation of continuity

$$\mathrm{div}(\gamma \vec{w}) = 0, \qquad (2.35)$$

the equation of motion (Darcy's law)

$$w = -\frac{k}{\mu}\frac{dp}{dr} \tag{2.36}$$

and the equation of state of a real gas

$$\frac{p}{\gamma} = Z(p, \overline{T}) R\overline{T}. \tag{2.37}$$

Combining these three equations, we obtain

$$\frac{\pi \bar{r} h k}{\mu G}\frac{dp^2}{dr} = Z(p, \overline{T})\overline{T}. \tag{2.38}$$

The system (2.33), (2.38) is solved on a computer using standard procedures. The fundamental difficulty connected with this is the calculation of the functions $\mu_i(p, \overline{T})$ and $Z(p, \overline{T})$. Attempts to select two-dimensional polynomials using tabulated data have so far failed to give satisfactory results [81]. There are also problems in introducing these tables into the memory of a computer [13].

It is probably most sensible from a practical point of view to use various semi-empirical equations of state. For example, in work [75] in analyzing the nonisothermal motion of a gas in a pipe, the Berthelot equation was used. In work [15], where the nonisothermal stationary flow of gas in a well was studied, an equation with three virial coefficients was used. In both cases the calculations

gave satisfactory results.

It is necessary, however, to remember that the use of any of the equations of state is valid only in some range of pressures and temperatures; moreover, the best method of verification is the construction, using the chosen equation, of the inversion curve corresponding to the case where the Joule-Thomson coefficient is equal to zero [81]. In particular, such a verification has shown that the applicability of the Berthelot equation is restricted to the value of the derived temperature $T_{der} = 1.3$, and an equation with virial coefficients in general poorly describes the inversion curve [81].

Section 3. The Effect of the Heat Field of Rock on the Thermal Regime of Wells

Earlier (Chapter 1 and Section 1 of Chapter 2) it was mentioned that the effect of nonstationary heat exchange between a flow in a well and rock may be taken into account using the method of successive replacement of stationary states [9, 19]. A different idea was enunciated by E. B. Chekalyuk [19]. He assumed in deriving an equation describing the nonstationary distribution of the temperature in a well that the flow of heat through the wall of the pipe may be written in the form

$$q = -\bar{\lambda}_1 \int_0^{\bar{t}} \alpha(\bar{t} - \tau) \frac{\partial \Delta \bar{T}}{\partial \tau} d\tau, \tag{2.39}$$

where $\alpha(\bar{t})$ is the time-dependent dimensionless heat exchange coefficient, and

$$\Delta\bar{T} = \bar{T} - \Theta_r$$

where the rock temperature Θ_r no longer depends on time.

A large number of works are known where various special cases of formula (2.39) are used. In the general case, in order to apply (2.39) it is necessary to know the form of the function $\alpha(\bar{t})$. In practice the formulas of I. A. Charnyy [9, 79] and E. B. Chekalyuk [19] are most often used; these formulas are obtained from an approximate solution of the corresponding heat conduction problems.

Below it will be shown that the intuitively introduced coefficient $\alpha(\bar{t})$ has a simple physical meaning. It is simplest to do this using the example of the operation of an oil well. Let the oil well operate with a constant yield Q. The temperature of the rock linearly increases with depth. For simplification of the calculations let us assume that the heat-transfer resistance of the walls of the borehole is negligibly small. Then the system of equations describing the given process, after a series of algebraic manipulations, has the form:

$$\frac{\partial\Delta\bar{T}}{\partial\tau} + \frac{\partial\Delta\bar{T}}{\partial\xi} = \beta\left(\frac{\partial\Delta\Theta}{\partial y}\right)_{y=1} + \frac{Q\Gamma}{\pi a_1}; \tag{2.40}$$

$$\frac{\partial \Delta\Theta}{\partial \tau} = \frac{\partial^2 \Delta\Theta}{\partial y^2} + \frac{1}{y}\frac{\partial \Delta Q}{\partial y}; \qquad (2.41)$$

$$\Delta \bar{T} = \Delta\Theta \quad \text{for} \quad y = 1; \qquad (2.42)$$

$$\Delta \bar{T} = \Delta\Theta = 0 \quad \text{for} \quad \tau = 0,\ \xi \geq 0; \qquad (2.43)$$

$$\Delta \bar{T} = 0 \quad \text{for} \quad \xi = 0,\ \tau > 0; \qquad (2.44)$$

$$\Delta\Theta \to 0 \quad \text{for} \quad y \to \infty. \qquad (2.45)$$

Here

$$\tau = a_1 \bar{t}/a^2;\ y = r/a;\ \xi = \pi a_1 x/Q;\ \beta = 2c_1\rho_1/c_p\rho;\ \Delta\bar{\Theta} = \bar{\Theta} - \Theta_r;$$

c_1, ρ_1 are the heat capacity and density of the rock; and Γ is

the geothermal gradient. The given boundary and initial conditions correspond to the equality of the temperatures in the well and in the rock before the well begins operating, and also to the equality of the stratum and bottom temperatures during the operation of the well.

From the solution of equation (2.41) with boundary conditions (2.42) and (2.45) and initial condition (2.43) and using the convolution theorem [82], we obtain

$$\left(\frac{\partial \Delta\Theta}{\partial y}\right)_{y=1} = -\frac{\partial}{\partial \tau}\int_0^{\tau} \varphi(\tau - z)\,\Delta\overline{T}(z)\,dz, \qquad (2.46)$$

where

$$\varphi(\tau) = \frac{4}{\pi^2}\int_0^{\infty} \frac{e^{-\tau u^2}}{J_0^2(u) + Y_0^2(u)} \cdot \frac{du}{u}; \qquad (2.47)$$

$J_0(u)$ and $Y_0(u)$ are Bessel functions of the first and second kind, of the zeroeth order. Substituting (2.47) into (2.40), we obtain

$$\frac{\partial \Delta\overline{T}}{\partial \tau} + \frac{\partial \Delta\overline{T}}{\partial \xi} + \beta\frac{\partial}{\partial \tau}\int_0^{\tau} \varphi(\tau - z)\,\Delta\overline{T}(z)\,dz - \frac{Q\Gamma}{\pi a_1} = 0. \qquad (2.48)$$

The function $\varphi(\tau)$ has a simple physical meaning. It is equal to the dimensionless heat flow at the wall of the borehole [83]. Using the Laplace transform one may show that for small values of time

$$\varphi(\tau) \approx 1/\sqrt{\pi\tau}, \qquad (2.49)$$

and then equation (2.48) coincides with the equation obtained by E. B. Chekalyuk for the flow of liquid in a plane channel [19]. This confirms the correctness of the result obtained here, since it is known [83] that for small values of time the solutions of axisymmetric and plane-parallel problems coincide.

Comparing the third term of equation (2.48) and formula (2.39), we convince ourselves that the function $\alpha(t)$ corresponds to the function $\varphi(\tau)$, i.e. this is also the dimensionless heat flow through a cylindrical surface over which is maintained a constant temperature.

In precisely the same way one may show that in the presence at the wall of the borehole of a heat-transfer resistance, i.e. for the replacement of boundary condition (2.42) by a boundary condition of the third kind, equation (2.48) retains its form, with the exception of the fact that for the function $\varphi(\tau)$ it is necessary to read in place of (2.47)

$$\varphi_1(\tau) = \frac{4}{\pi^2}\,\mathrm{Nu}\int_0^\infty \frac{e^{-\tau u^2}\,du}{u\,\{[uJ_1(u) + \mathrm{Nu}\,J_0(u)]^2 + [uY_1(u) + \mathrm{Nu}\,Y_0(u)]^2\}}, \qquad (2.50)$$

where $J_1(u)$ and $Y_1(u)$ are Bessel functions of the first and second kind, of the first order, and

$$\mathrm{Nu} = \frac{\bar{\alpha} a}{\bar{\lambda}_1}$$

is the Nusselt number.

It is most convenient to find the solution of the various boundary-value problems for equation (2.48) using the Laplace transform. As an example, let us examine the solution of this equation for boundary condition (2.44) and initial condition (2.43). Using the Laplace transform

$$w = \int_0^\infty e^{-p\tau} \Delta \overline{T}(\tau)\, d\tau$$

in place of (2.43), (2.44), and (2.48), we obtain

$$\frac{dw}{d\xi} + \left[p + \beta\overline{\varphi}(p)\right] w - \frac{Q\Gamma}{\pi a_1 p} = 0, \tag{2.51}$$

$$\omega(0) = 0, \tag{2.52}$$

where p is the transform parameter;

$$\overline{\varphi}(p) = \int_0^\infty e^{-p\tau} \varphi(\tau)\, d\tau. \tag{2.53}$$

The solution of equation (2.51) for boundary condition (2.52) has the form

$$w = \frac{Q\Gamma}{\pi a_1 \left[p + \beta\overline{\varphi}(p)\right] p} \left\{1 - e^{-[p+\beta\overline{\varphi}(p)]\xi}\right\}. \tag{2.54}$$

In the general case, in order to pass from the representation of w to the original quantity $\Delta \bar{T}$ it is necessary to use one of the numerical methods of [82], as is done in work [11] in calculating the thermal regime of bored wells. The solutions for small τ may be found by using the asymptotic function $\bar{\varphi}(p)$, and for large values of time the solutions may be found using the method presented in [83].

For many practically interesting cases the method of quasi-stationary states assures sufficient accuracy. In accordance with this method, the solution of the just-examined problem will have the form

$$\Delta \bar{T} = \frac{Q\Gamma}{\pi a_1 \beta \alpha(\tau)} [1 - e^{-\alpha(\tau)\xi}], \qquad (2.55)$$

where according to [19]

$$\alpha(\tau) = \frac{2\pi}{\ln(1 + \sqrt{\pi\tau})}. \qquad (2.56)$$

If the temperature at the bottom of the borehole is less than the stratum temperature due to orificing, then the corresponding solution has the form

$$\Delta \bar{T} = \Delta \bar{T}_з e^{-\alpha(\tau)\xi} + \frac{Q\Gamma}{\pi a_1 \alpha(\tau)} [1 - e^{-\alpha(\tau)\xi}]. \qquad (2.57)$$

From (2.57) one may find the depth at which the gas that is cooled as a result of orificing is heated up to the temperature of the rock at the given point. Setting the left hand side equal to zero, we obtain after straightforward algebra

$$\xi = \frac{1}{\alpha} \ln\left(1 - \frac{\pi a_1 \alpha \Delta \bar{T}_3}{\Gamma Q}\right). \tag{2.58}$$

For the majority of practically interesting cases the second term under the logarithm sign is on the order of 10^{-3}. Then from (2.58) it follows that

$$\xi \approx -\frac{\pi a_1 \Delta \bar{T}_3}{\Gamma Q}.$$

Taking into account the expression for ξ, we finally obtain

$$x \approx -\frac{\Delta \bar{T}_3}{\Gamma}, \tag{2.59}$$

where x is the distance from the bottom. A taking into account of the real properties of the gas during its motion in the well leads to an increase in this distance.

Section 4. Formulation of the Problem of the Heat Exchange Between Frozen Rock and a Well

Let us examine the thermodynamical regime of exploitation of a well in frozen ground (Fig. 1). It is necessary to note that strata of frozen ground practically never attain the designation of

producing beds, having, as a rule, a very high positive temperature. Consequently, the cross section of a well in a region of permafrost may be presented in the following manner: the producing bed, a stratum of warm (unfrozen) ground, and a stratum of frozen ground. During the motion of gas (oil) in the well through the frozen zone there occurs, on the one hand, a lowering of the temperature of the gas (oil) due to the intensive transfer of heat into the frozen ground, and on the other hand, the thermal influence of the well leads to the thawing of the frozen ground located next to the well. The thawing boundary shifts with the passage of time, which entails a change in the coefficient of heat transfer from the gas (oil) in the well to the ground.

Thus, there arises a nonstationary thermal regime in the well and in the stratum. Since the methodology for calculating the thermal regime of a well in unfrozen ground is elucidated in Section 3, we shall examine only that part of the well which passes through the region of frozen ground, considering the temperature of the gas (oil) at the entrance to the frozen stratum to be known.

In order to derive the fundamental equations describing the temperature fields in the well and in the ground surrounding it, let us make the following assumptions:

1) all the phase transformations in the frozen ground take place at 0°C;

2) the vertical heat flows in the ground are small in comparison with the radial flows;

3) the vertical heat flow in the well due to heat conduction and the heat flow due to the dissipation of mechanical energy are small in comparison with the convective heat flow.

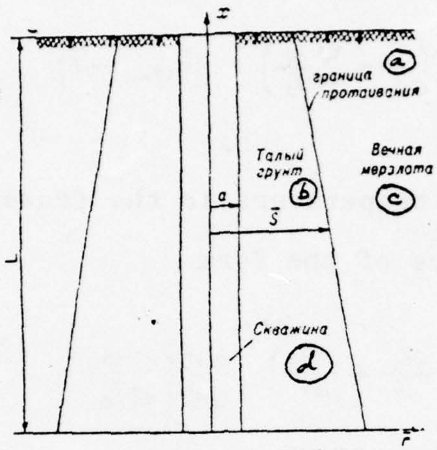

Fig. 1.

Key: a - thawing boundary
b - thawed ground
c - permafrost
d - borehole

Taking into account these assumptions let us write down the equation of heat conduction for the frozen and thawed regions in the ground. The heat conduction equation for the thawed region has the form

$$\frac{\partial\overline{\Theta}_1}{\partial\bar{t}} = a_1\left(\frac{1}{\bar{r}}\frac{\partial\overline{\Theta}_1}{\partial\bar{r}} + \frac{\partial^2\overline{\Theta}_1}{\partial\bar{r}^2}\right); \quad \begin{matrix} a < \bar{r} < \bar{S}(\bar{t}, x) \\ \bar{t}_m \leqslant \bar{t} \leqslant \infty \end{matrix} \qquad (2.60)$$

The boundary conditions are as follows:

$$\bar{\Theta}_1\big|_{\bar{r}=\bar{S}} = 0; \tag{2.61}$$

$$\bar{\lambda}_1 \frac{\partial \bar{\Theta}_1}{\partial \bar{r}}\bigg|_{\bar{r}=a} = \left(\frac{1}{\bar{\alpha}} + \sum_i \frac{\delta_i}{\bar{\lambda}_i}\right)^{-1} \left(\bar{\Theta}_1\big|_{\bar{r}=a} - \bar{T}\right). \tag{2.62}$$

The equations for the temperature in the frozen region up to the beginning of thawing are of the form

$$\frac{\partial \bar{\Theta}_2}{\partial \bar{t}} = a_2\left(\frac{1}{\bar{r}}\frac{\partial \bar{\Theta}_2}{\partial \bar{r}} + \frac{\partial^2 \bar{\Theta}_2}{\partial \bar{r}^2}\right); \quad \begin{matrix} a < \bar{r} < \infty \\ 0 \leq \bar{t} \leq \bar{t}_m \end{matrix} \tag{2.63}$$

The boundary conditions are as follows:

$$\bar{\lambda}_2 \frac{\partial \bar{\Theta}_2}{\partial \bar{r}}\bigg|_{\bar{r}=a} = \left(\frac{1}{\bar{\alpha}} + \sum_i \frac{\delta_i}{\bar{\lambda}_i}\right)^{-1} \left(\bar{\Theta}_2\big|_{\bar{r}=a} - \bar{T}\right); \tag{2.64}$$

$$\bar{\Theta}_2\big|_{\bar{t}=0} = \bar{\Theta}_M \leq 0. \tag{2.65}$$

The equation for the temperature in the frozen region after the beginning of thawing has the form

$$\frac{\partial \bar{\Theta}_2}{\partial \bar{t}} = a_2\left(\frac{1}{\bar{r}}\frac{\partial \bar{\Theta}_2}{\partial \bar{r}} + \frac{\partial^2 \bar{\Theta}_2}{\partial \bar{r}^2}\right); \quad \begin{matrix} \bar{S}(\bar{t}, x) < \bar{r} < \infty \\ \bar{t}_m \leq \bar{t} < \infty \end{matrix} \tag{2.66}$$

The boundary conditions are as follows:

$$\bar{\Theta}_2|_{\bar{r}=\bar{r}_m} = \bar{\Theta}_0(\bar{r}, x); \tag{2.67}$$

$$\bar{\Theta}_2|_{\bar{r}=\bar{S}} = 0. \tag{2.68}$$

The condition at the thawing boundary (Stefan's condition) is as follows:

$$-\bar{\lambda}_1 \frac{\partial \bar{\Theta}_1}{\partial \bar{r}} + \bar{\lambda}_2 \frac{\partial \bar{\Theta}_2}{\partial \bar{r}} = l\rho_T \frac{\partial \bar{S}}{\partial \bar{t}} \quad \text{for } \bar{r} = \bar{S} \tag{2.69}$$

The heat flow equation for the gas (oil) in the well (2.25) under the assumption of a linear distribution of the pressure over the height may be written in the following form:

$$\rho f c_p \frac{\partial \bar{T}}{\partial \bar{t}} + c_p \dot{M} \frac{\partial \bar{T}}{\partial x} + \frac{c_p M K (p_1 - p_2)}{L} = 2\pi a \bar{\lambda}_1 \frac{\partial \bar{\Theta}_1}{\partial \bar{r}}\Big|_{\bar{r}=a}. \tag{2.70}$$

Here

$$K = \mu_i + \frac{A}{\rho c_p}$$

for gas and k = 0 for oil.

We shall consider that at the initial moment in time $\bar{t} = 0$,

gas (oil) having in the initial section x = 0 a temperature $\bar{T}_0$ begins to fill the well, moving with a velocity v.

Thus, the region of the change in the independent variables x and $\bar{t}$ is determined in the following manner:

$$0 \leqslant x \leqslant v\bar{t};$$
$$0 \leqslant x \leqslant L.$$

The boundary conditions are as follows:

$$\bar{T}|_{x=0} = \bar{T}_0(\bar{t}). \tag{2.71}$$

Let us write problem (2.60)-(2.71) in dimensionless form. In order to do this let us introduce the following dimensionless variables:

$$\Theta_1 = \frac{\bar{\Theta}_1}{T_{\text{пл}}};\quad \Theta_2 = \frac{\bar{\Theta}_2}{T_{\text{пл}}};\quad t = \frac{\bar{t}}{t_0};\quad r = \frac{\bar{r}}{a};\quad S = \frac{\bar{S}}{a};\quad T_{\text{пл}} \equiv T$$
$$z = \frac{x}{L};\quad T = \frac{\bar{T}}{T_{\text{пл}}};\quad t_m = \frac{\bar{t}_m}{t_0};\quad T_0 = \frac{\bar{T}_0}{T_{\text{пл}}};\ \Theta_{\text{м}} = \frac{\bar{\Theta}_{\text{м}}}{T_{\text{пл}}}\quad \Theta_M \equiv \Theta_F$$

Substituting these variables into the equations and boundary conditions (2.60)-(2.71), we obtain the following system of dimensionless equations determining the thermal process in the well and the surrounding ground. For the thawed region we have:

$$\frac{\partial\Theta_1}{\partial t} = \varkappa_1\left(\frac{1}{r}\cdot\frac{\partial\Theta_1}{\partial r} + \frac{\partial^2\Theta_1}{\partial r^2}\right);\qquad \begin{matrix} 1 < r < S(t, z) \\ t_m < t < \infty \end{matrix} \tag{2.72}$$

$$\Theta_1|_{r=S} = 0; \tag{2.73}$$

$$\left.\frac{\partial\Theta_1}{\partial r}\right|_{r=1} = \alpha\,(\Theta_1|_{r=1} - T), \tag{2.74}$$

where

$$\alpha = \frac{a}{\bar{\lambda}_1}\left(\frac{1}{\bar{\alpha}} + \sum_i \frac{\delta_i}{\bar{\lambda}_i}\right)^{-1}.$$

For the frozen region and for $t \leq t_m$,

$$\frac{\partial\Theta_2}{\partial t} = \varkappa_2\left(\frac{1}{r}\frac{\partial\Theta_2}{\partial r} + \frac{\partial^2\Theta_2}{\partial r^2}\right); \quad \begin{matrix} 1 < r < \infty \\ 0 \leqslant t \leqslant t_m \end{matrix} \tag{2.75}$$

$$\Theta_2|_{t=0} = \Theta_F < 0; \tag{2.76}$$

$$\left.\frac{\partial\Theta_2}{\partial r}\right|_{r=1} = \alpha_2\,(\Theta_2|_{r=1} - T), \tag{2.77}$$

where

$$\alpha_2 = \frac{a}{\lambda_2}\left(\frac{1}{\alpha} + \sum \frac{\delta_i}{\lambda_i}\right)^{-1}.$$

For the frozen region and for $t \geqslant t_m$,

$$\frac{\partial \Theta_2}{\partial t} = \varkappa_2\left(\frac{1}{r}\frac{\partial \Theta_2}{\partial r} + \frac{\partial^2 \Theta_2}{\partial r^2}\right); \quad \begin{array}{l} S(t, z) < r < \infty \\ t_m \leqslant t < \infty \end{array} \tag{2.78}$$

$$\Theta_2\Big|_{t=t_m} = \Theta_0(r, z); \tag{2.79}$$

$$\Theta_2\Big|_{r=S} = 0. \tag{2.80}$$

Stefan's condition is as follows:

$$-\lambda_1 \frac{\partial \Theta_1}{\partial r} + \lambda_2 \frac{\partial \Theta_2}{\partial r} = \frac{\partial S}{\partial t} \quad \text{for } r = S \tag{2.81}$$

$$S\Big|_{t=t_m} = 1. \tag{2.82}$$

The heat flow equation for the gas (oil) is

$$C_1 \frac{\partial T}{\partial \tau} + C_2 \frac{\partial T}{\partial z} = \frac{\partial \Theta_1}{\partial r}\Big|_{r=1} - C_3; \tag{2.83}$$

$$T|_{z=0} = T_0(t). \tag{2.84}$$

Here

$$C_1 = \frac{\rho c_p f}{2\bar{\lambda}_1 t_0 \pi}; \quad C_2 = \frac{c_p M}{2\pi L \bar{\lambda}_1}; \quad C_3 = \frac{c_p M K (p_1 - p_2)}{2\pi \bar{\lambda}_1 T_{\text{пл}} L};$$

$$\varkappa_i = \frac{a_i t_0}{a^2}; \quad \lambda_i = \frac{\bar{\lambda}_i T_{\text{пл}} t_0}{\rho_T l a^2} \quad i = 1; 2. \quad T_{\text{пл}} \equiv T_{st}$$

CHAPTER 3
THERMAL REGIME OF WELLS IN FROZEN GROUND

Section 1. Approximate Solution of the Plane Problem of the Thawing of Frozen Ground Around a Well

For high-yield wells the temperature of gas (oil) varies little with time. Indeed, with an increase in M equation (2.83) approaches the form

$$C_2 \frac{\partial T}{\partial z} = -C_3.$$

The solution of this equation does not depend on time, and for each horizontal section of the stratum it may be considered to be constant, but varying from section to section, and one may consider in each section the plane Stefan problem. Even for such a simplified formulation, in order to obtain an exact solution of this problem it is necessary to enlist the aid of numerical methods, which will be examined in the Appendix. Realization of these methods is connected with the writing of very complex programs for a computer. At the same time, with an accuracy sufficient for engineering calculations, this problem may be solved using the method of successive approximations described in Chapter 1. Using this method one will obtain a solution directly in the form of quadratures for the case where the frozen ground is at the thawing temperature. In the more general case, where the initial temperature of the frozen ground is below

zero, the problem reduces to a system of two ordinary differential equations, the solution of which is not difficult to find, in particular when one uses standard programs for a computer.

Let us examine in detail both of the indicated cases and compare the results with the exact solution.

Let us consider the application of the indicated method to the solution of problem (2.72)-(2.81). In order to do this, in conditions (2.74) and (2.77) let us assume that T = 1. For the thawed region the zeroeth approximation is found from the equation

$$\frac{1}{r}\cdot\frac{\partial\Theta_1^{(0)}}{\partial r}+\frac{\partial^2\Theta_1^{(0)}}{\partial r^2}=0. \tag{3.1}$$

The boundary conditions are as follows:

$$\Theta_1^{(0)}\big|_{r=S}=0; \tag{3.2}$$

$$\left.\frac{\partial\Theta_1^{(0)}}{\partial r}\right|_{r=1}=\alpha\left(\Theta_1^{(0)}\big|_{r=1}-1\right). \tag{3.3}$$

Integrating equation (3.1) twice, we obtain

$$\Theta_1^{(0)}=C_1\ln r+C_2. \tag{3.4}$$

Substituting here the boundary conditions (3.2) and (3.3), we obtain

$$\Theta_1^{(0)} = -\frac{\alpha}{\alpha \ln S + 1} \ln \frac{r}{S}. \qquad (3.5)$$

We find the first approximation from the equation

$$\varkappa_1 \left(\frac{1}{r} \cdot \frac{\partial \Theta_1^{(1)}}{\partial r} + \frac{\partial^2 \Theta_1^{(1)}}{\partial r^2} \right) = \frac{\partial \Theta_1^{(0)}}{\partial t}. \qquad (3.6)$$

The boundary conditions are as follows:

$$\Theta_1^{(1)}|_{r=S} = 0; \qquad (3.7)$$

$$\left. \frac{\partial \Theta_1^{(1)}}{\partial r} \right|_{r=1} = \alpha \Theta_1^{(1)}|_{r=1}. \qquad (3.8)$$

Substituting into (3.6) the expression (3.5) for $\Theta_1^{(0)}$, we obtain the equation

$$\varkappa_1 \left(\frac{1}{r} \frac{\partial \Theta_1^{(1)}}{\partial r} + \frac{\partial^2 \Theta_1^{(1)}}{\partial r^2} \right) \equiv \frac{\varkappa_1}{r} \cdot \frac{\partial}{\partial r} \left(r \frac{\partial \Theta_1^{(1)}}{\partial r} \right) = \frac{\alpha \dot{S} (1 + \alpha \ln r)}{S (\alpha \ln S + 1)^2}. \qquad (3.9)$$

Here $\dot{S} \equiv \frac{\partial S}{\partial t}$. Integrating the last equation, we obtain:

$$\frac{\partial \Theta_1^{(1)}}{\partial r} = \frac{\alpha \dot{S}}{2\varkappa_1 S (\alpha \ln S + 1)^2} \left(r + \alpha r \ln r - \frac{\alpha r}{2} \right) + \frac{C_1}{r}; \qquad (3.10)$$

$$\Theta_1^{(1)} = \frac{\alpha \dot{S} r_2 (1 - \alpha + \alpha \ln r)}{4\varkappa_1 S (\alpha \ln S + 1)^2} + C_1 \ln r + C_2. \tag{3.11}$$

The constants C_1 and C_2 are obtained from boundary conditions (3.7) and (3.8).

Restricting ourselves to two approximations, we may write

$$\Theta_1 \approx \Theta_1^{(0)} + \Theta_1^{(1)}. \tag{3.12}$$

Obviously, the function of (3.12) satisfies boundary conditions (2.73)-(2.74). From this we find

$$\left.\frac{\partial \Theta_1}{\partial r}\right|_{r=S} = \frac{\dot{S}\alpha}{4(\alpha \ln S + 1)^3 \varkappa_1}\Big[2(\alpha \ln S + 1)^2 - \\ - 2\alpha(\alpha \ln S + 1) + \alpha^2 - \frac{2 - 2\alpha + \alpha^2}{S^2}\Big] - \frac{\alpha}{S(\alpha \ln S + 1)}. \tag{3.13}$$

Analogously, for the frozen region we look for the zeroeth approximation from the equation

$$\frac{1}{r}\frac{\partial \Theta_2^{(0)}}{\partial r} + \frac{\partial^2 \Theta_2^{(0)}}{\partial r^2} = 0. \tag{3.14}$$

In view of the fact that the condition

$$\Theta_2 \rightarrow \Theta_F \quad \text{as} \quad r \rightarrow \infty,$$

which follows immediately from (2.79), cannot be satisfied by any

solution of equation (3.14), let us introduce the radius of thermal influence R(t) [9], over which we take the following conditions:

$$\Theta_2\big|_{r=R} = \Theta_F \,; \tag{3.15}$$

$$\frac{\partial \Theta_2}{\partial r}\bigg|_{r=R} = 0. \tag{3.16}$$

Let us write the boundary conditions for equation (3.14) in the following form:

$$\Theta_2^{(0)}\big|_{r=S} = 0 \,; \tag{3.17}$$

$$\Theta_2^{(0)}\big|_{r=R} = \Theta_F \,. \tag{3.18}$$

Integrating (3.14) taking into account (3.17) and (3.18), we obtain:

$$\frac{\partial \theta_2^{(0)}}{\partial r} = \frac{\theta_M}{r(\ln R - \ln S)}; \quad \Theta_M \equiv \Theta_F \tag{3.19}$$

$$\Theta_2^{(0)} = \frac{\Theta_M(\ln r - \ln S)}{\ln R - \ln S}. \quad \Theta_M \equiv \Theta_F \tag{3.20}$$

We find the first approximation from the equation

$$\varkappa_2\left(\frac{1}{r}\cdot\frac{\partial\Theta_2^{(1)}}{\partial r}+\frac{\partial^2\Theta_2^{(1)}}{\partial r^2}\right)=\frac{\partial\Theta_2^{(0)}}{\partial t}. \tag{3.21}$$

The boundary conditions are as follows:

$$\Theta_2^{(1)}\Big|_{r=S}=0; \tag{3.22}$$

$$\Theta_2^{(1)}\Big|_{r=R}=0. \tag{3.23}$$

Substituting into (3.21) expression (3.20), we obtain the equation

$$\frac{\varkappa_2}{r}\cdot\frac{\partial}{\partial r}\left(r\frac{\partial\Theta_2^{(1)}}{\partial r}\right)=\Theta_\text{M}\frac{\left[\ln r\left(\frac{\dot S}{S}-\frac{\dot R}{R}\right)+\frac{\dot R}{R}\ln S-\frac{\dot S}{S}\ln R\right]}{(\ln R-\ln S)^2}. \tag{3.24}$$

Integrating this equation twice, we obtain:

$$\frac{\partial\Theta_2^{(1)}}{\partial r}=\frac{\Theta_\text{M}}{2(\ln R-\ln S)^2\varkappa_2}\left[\left(r\ln r-\frac{r}{2}\right)\left(\frac{\dot S}{S}-\frac{\dot R}{R}\right)+\right. \\ \left.+r\left(\frac{\dot R}{R}\ln S-\frac{\dot S}{S}\ln R\right)\right]+\frac{C_1}{r}; \tag{3.25}$$

$$\Theta_2^{(1)}=\frac{\Theta_\text{M}}{4\varkappa_2(\ln R-\ln S)^2}\left[(r^2\ln r-r^2)\left(\frac{\dot S}{S}-\frac{\dot R}{R}\right)+\right. \\ \left.+r^2\left(\frac{\dot R}{R}\ln S-\frac{\dot S}{S}\ln R\right)\right]+C_1\ln r+C_2. \tag{3.26}$$

By satisfying boundary conditions (3.22) and (3.23) we determine C_1 and C_2.

Restricting ourselves to two approximations, we write:

$$\Theta_2 \approx \Theta_2^{(0)} + \Theta_2^{(1)}. \tag{3.27}$$

The function of (3.27) satisfies the boundary conditions

$$\Theta_2\Big|_{r=S} = 0, \tag{3.28}$$

$$\Theta_2\Big|_{r=R} = \Theta_F. \tag{3.29}$$

From (3.27) we find, taking into account (3.19) and (3.25),

$$\left.\frac{\partial\Theta_2}{\partial r}\right|_{r=S} = \frac{\dot{R}\Theta_M\left[(R^2+S^2)(\ln R-\ln S)-R^2+S^2\right]}{4\varkappa_2 RS(\ln R-\ln S)^3} + \qquad \Theta_M \equiv \Theta_F$$

$$+\frac{\dot{S}\Theta_M\left[2(\ln R-\ln S)(\ln S-\ln R-1)+R^2/S^2-1\right]}{4\varkappa_2(\ln R-\ln S)^3} + \frac{\Theta_M}{S(\ln R-\ln S)}. \tag{3.30}$$

Hence Stefan's condition (2.81), taking into account (3.13) and (3.30), acquires the form

$$\dot{S}\left\{\frac{\lambda_2\Theta_M\left[2(\ln R-\ln S)(\ln S-\ln R-1)+R^2/S^2-1\right]}{4\varkappa_2(\ln R-\ln S)^3} - \qquad \Theta_M \equiv \Theta_F\right.$$

$$-\frac{\lambda_1\alpha}{4\varkappa_1(\alpha\ln S+1)^3}\Big[2(\alpha\ln S+1)^2-2\alpha(\alpha\ln S+1)+\alpha^2- \tag{3.31}$$

$$\left.-\frac{2-2\alpha+\alpha^2}{S^2}\Big]-1\right\} + \dot{R}\frac{\lambda_2\Theta_M\left[(R^2+S^2)(\ln R-\ln S)-R^2+S^2\right]}{4\varkappa_2 RS(\ln R-\ln S)^3} =$$

$$= -\frac{\lambda_2\Theta_M}{S(\ln R-\ln S)} - \frac{\lambda_1\alpha}{S(\alpha\ln S+1)}.$$

In order to derive the second equation let us use condition (3.16). Taking into account (3.27) we obtain

$$\dot{R}[2(\ln R - \ln S)(\ln S - \ln R + 1) - 1 + S^2/R^2] + \\ + \dot{S}\left[\frac{R^2 - S^2 - (R^2 + S^2)(\ln R - \ln S)}{RS}\right] = -\frac{4\varkappa_2(\ln R - \ln S)^2}{R}. \tag{3.32}$$

The problem reduces to the solution of a system of two ordinary differential equations of the first order relative to the unknown functions S(t) and R(t) of (3.31) and (3.32). The boundary conditions of the system of equations (3.31)-(3.32) have the form:

$$R = R_m \quad \text{for} \quad t = t_m \tag{3.33}$$

$$S = 1 \quad \text{for} \quad t = t_m \tag{3.34}$$

Here R_m is the value of the radius of thermal influence at the moment thawing begins. Let us find R_m and t_m from examining problem (2.75)-(2.77) regarding the heating of frozen ground until the ground at the wall of the borehole begins to thaw.

Analogous to what was done previously, we look for the zeroeth approximation as a solution of equation (3.14) with boundary conditions

$$\left.\frac{\partial \Theta_2^{(0)}}{\partial r}\right|_{r=1} = \alpha\left(\Theta_2^{(0)}|_{r=1} - 1\right); \tag{3.35}$$

$$\Theta_2^{(0)}\Big|_{r=R} = \Theta_F . \tag{3.36}$$

The solution of this problem has the form:

$$\frac{\partial\Theta_2^{(0)}}{\partial r} = \frac{\alpha_2(\Theta_M - 1)}{r(\alpha_2 \ln R + 1)}, \qquad \Theta_M \equiv \Theta_F \tag{3.37}$$

$$\Theta_2^{(0)} = \frac{\alpha_2(\Theta_M - 1)\ln r + \Theta_M + \alpha_2 \ln R}{\alpha_2 \ln R + 1}. \qquad \Theta_M \equiv \Theta_F \tag{3.38}$$

Let us look for the first approximation as a solution of equation (3.21) with boundary conditions:

$$\Theta_2^{(1)}\big|_{r=R} = 0, \tag{3.39}$$

$$\frac{\partial\Theta_2^{(1)}}{\partial r}\Big|_{r=1} = \alpha_2\Theta_2^{(1)}. \tag{3.40}$$

The solution of this problem has the form:

$$\frac{\partial\Theta_2^{(1)}}{\partial r} = \frac{\alpha_2\dot{R}(1-\Theta_M)(r + \alpha_2 r \ln r - \alpha_2 r/2)}{2\varkappa_2 R(\alpha_2 \ln R + 1)^2} + \frac{C_1}{r}; \tag{3.41}$$

$$\Theta_2^{(1)} = \frac{\alpha_2 \dot{R}(1-\Theta_M) r^2 (1-\alpha_2+\alpha_2 \ln r)}{4\varkappa_2 R(\alpha_2 \ln R+1)^2} + C_1 \ln r + C_2. \tag{3.42}$$

The constants of integration C_1 and C_2 are determined from the boundary conditions (3.39) and (3.40).

Restricting ourselves to two approximations, we assume

$$\Theta_2 \approx \Theta_2^{(0)} + \Theta_2^{(1)}. \tag{3.43}$$

Obviously, the function of (3.43) satisfies boundary conditions (2.77) and (3.15).

Satisfying condition (3.16), we obtain, taking into account (3.37) and (3.41),

$$\dot{R} = \frac{4\varkappa_2(\alpha_2 \ln R+1)^3}{R\left[2(\alpha_2 \ln R+1)^2 - 2\alpha_2(\alpha_2 \ln R+1) + \alpha_2^2 - \frac{2-2\alpha_2+\alpha_2^2}{R^2}\right]}. \tag{3.44}$$

Hence, satisfying the boundary condition R = 1 at t = 0, we obtain

$$t = \frac{R^2}{4\varkappa_2} - \frac{R_2}{4\varkappa_2 \ln(Re^{1/\alpha_2})} + \frac{2-2\alpha_2+\alpha_2^2}{4\varkappa_2\alpha_2^2 \ln(R\cdot e^{1/\alpha_2})} + \frac{\alpha_2-2}{4\varkappa_2\alpha_2}. \tag{3.45}$$

Setting equal to zero the expression for the temperature of the wall of the borehole Θ_2 at r = 1, we obtain the equation

$$\dot{R}\left[(2-\alpha_2)\ln R + 1 - \alpha_2 - R^2(1+\alpha_2 \ln R - \alpha_2)\right] + \frac{4(\alpha_2 \ln R+1)^2 \varkappa_2 R \ln R}{1-\Theta_M} + \frac{4\Theta_M \varkappa_2 R(\alpha_2 \ln R+1)^2}{\alpha_2(1-\Theta_M)} = 0. \tag{3.46}$$

$\Theta_M \equiv \Theta_F$

Substituting into this the expression for R from (3.44), we determine R_m, and substituting its value into (3.45), we obtain the value of t_m.

In view of the fact that the temperature of frozen rock in the majority of cases is close to the thawing temperature (0°C to -2°C), it is of interest to examine the solution of the given problem for $\Theta_F = 0$. In this case equation (3.31) acquires the form

$$\dot{S}\left\{-\frac{\lambda_1\alpha}{4\varkappa_1(\alpha\ln S+1)^3}\left[2(\alpha\ln S+1)^2-2\alpha(\alpha\ln S+1)+\right.\right.$$
$$\left.\left.+\alpha^2-\frac{2-2\alpha+\alpha^2}{S^2}\right]-1\right\}=-\frac{\lambda_1\alpha}{S(\alpha\ln S+1)}. \qquad (3.47)$$

Its solution, satisfying the boundary condition S = 1 at t = 0, has the form

$$t=\frac{S^2}{4\varkappa_1}\left[1-\frac{1}{\ln(Se^{1/\alpha})}\right]+\frac{2-2\alpha+\alpha^2}{4\varkappa_1\alpha^2\ln(Se^{1/\alpha})}+$$
$$+\frac{S^2[2\ln(Se^{1/\alpha})-1]}{4\lambda_1}+\frac{\alpha-2}{4\alpha}\left(\frac{1}{\varkappa_1}+\frac{1}{\lambda_1}\right). \qquad (3.48)$$

Let us calculate the rate of thawing of the frozen ground around a gas well. Let us take the characteristics of the ground from data for the Urengoy deposit. Let us make the calculation for the following data:

$a_1 = 2.6\times10^{-3}$ m²/hour; $a_2 = 3.05\times10^{-3}$ m²/hour;

$\rho_T = 292$ kg/m³; $a = 0.14$ m; $T = 25$°C.; $t_0 = 2$ hours.

The dimensionless parameters introduced above will have the values:

$$\lambda_1 = 0.168; \quad \lambda_2 = 0.196; \quad \varkappa_1 = 0.231; \quad \varkappa_2 = 0.271.$$

The dimensionless heat transfer coefficients take on the following values:

$$\alpha = 0.55; \quad \alpha_2 = 0.472$$

The result of calculating the rate of thawing of frozen ground with a zero initial temperature using formula (3.48) is presented in Fig. 2 by a solid line. The dotted line shows the exact solution of the problem, obtained by numerical methods using a computer. From a comparison of the curves it is apparent that the error in the approximation method in question under the conditions of the given problem is very small. For example, for t = 40 the value of S for the exact solution is 2.502, and for the approximate one, 2.487. The error estimated using the formula

$$\delta = \frac{\Delta S}{S-1} \cdot 100\%$$

is about 1%.

This figure shows, by means of a stroke-dotted line, the solution obtained by the method of successive replacement of stationary

states [54]. The error in this method for t = 40 is about 4%.

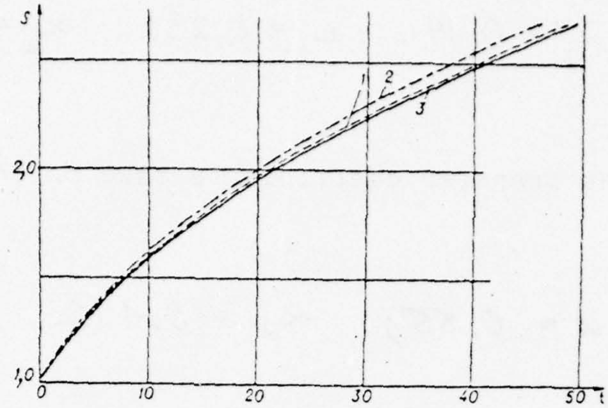

Fig. 2.

1 - exact solution; 2 - quasistationary solution;
3 - solution using formula (3.48).

Let us return to the general case of a negative initial temperature of the frozen ground. It is obvious that system (3.31)-(3.32) does not admit a solution in analytic form. Therefore, the solution was obtained on a "Nairi" computer with the use of a standard program for integrating a system of ordinary differential equations using the Runge-Kutta method. In order to accomplish this, system of equations (3.31)-(3.32) was solved for $\dot{R}$ and $\dot{S}$:

$$\dot{S} = \left[\frac{\lambda_2\theta_{\text{м}}}{S}\left(2\ln\frac{R}{S} + \frac{S^2}{R^2} - 1\right) + \frac{\lambda_1\alpha\left(2\ln^2\frac{R}{S} - 2\ln\frac{R}{S} + 1 - \frac{S^2}{R^2}\right)}{S(\alpha\ln S + 1)}\right] \Bigg/ \left\{\frac{\lambda_2\theta_{\text{м}}\left(4\ln^2\frac{R}{S} + 2 - \frac{R^2}{S^2} - \frac{S^2}{R^2}\right)}{4\varkappa_2\ln\frac{R}{S}} - \left[\frac{S^2}{R^2} - 1 + 2\ln\frac{R}{S}\left(1 - \frac{R}{S}\right)\right] \times \right.$$
$$\left. \times \left[\frac{\lambda_1\alpha\left(2(\alpha\ln S + 1)^2 - 2\alpha(\alpha\ln S + 1) + \alpha^2 - \frac{2 - 2\alpha + \alpha^2}{S^2}\right)}{4\varkappa_1(\alpha\ln S + 1)^3} + 1\right]\right\}; \qquad (3.49)$$

$\Theta_M \equiv \Theta_F$

$$
\begin{aligned}
\dot{R} = \Bigg\{ & \frac{\lambda_2\theta_M\left(R^2 - S^2 - 2S^2 \ln \frac{R}{S}\right)}{RS^2} + \frac{\lambda_1\alpha}{(\alpha \ln S + 1)} \times \\
& \times \left[\frac{\left(S^2 - R^2 + (R^2+S^2) \ln \frac{R}{S}\right)}{RS^2} - \frac{\varkappa_2 \ln^2 \frac{R}{S}}{\varkappa_1 R (\alpha \ln S + 1)^2} \Big(2(\alpha \ln S + 1)^2 - \right. \\
& \left. - 2\alpha(\alpha \ln S + 1) + \alpha^2 - \frac{2 - 2\alpha + \alpha^2}{S^2} \Big) \right] - \frac{4\varkappa_2 \ln^2 \frac{R}{S}}{R} \Bigg\} \Bigg/ \\
& \Bigg/ \Bigg\{ \left[\frac{S^2}{R^2} - 1 + 2 \ln \frac{R}{S} \left(1 - \ln \frac{R}{S}\right) \right] \times \\
& \times \left[\frac{\lambda_1\alpha\left(2(\alpha \ln S + 1)^2 - 2\alpha(\alpha \ln S + 1) + \alpha^2 - \frac{2 - 2\alpha + \alpha^2}{S^2}\right)}{4\varkappa_1 (\alpha \ln S + 1)^3} + 1 \right] - \\
& - \frac{\lambda_2\theta_M\left(4 \ln^2 \frac{R}{S} + 2 - R^2/S^2 - S^2/R^2\right)}{4\varkappa_2 \ln R/S} \Bigg\};
\end{aligned}
\qquad (3.50)
$$

$\Theta_M \equiv \Theta_F$

The calculation was made using the values of the dimensionless parameters indicated above, and for the following values of the dimensionless initial temperatures:

$$\Theta_F = -0.1;\ -0.25;\ -0.5.$$

Here R_m had the values 1.432, 2.285, and 4.220 respectively, and t_m had the values 0.285, 2.0, and 10.31.

The results of the calculation are presented in Fig. 3 (solid lines). The dotted line shows the results of the exact solution of problem (2.72)-(2.81) for Θ_F = -0.1, obtained by reduction to a system of integral equations. As is apparent from comparison of these results, the approximation method used gives very good approximations to the exact solution. Thus, for t = 40 the value for the exact solution is 2.167, and for the approximate solution,

the value is 2.175. The error is about 1%.

In order to obtain an overall appraisal of the accuracy of the method of successive approximations in solving Stefan's problem we made a comparison of the solution, found using the indicated method, of the problem of the advancement of a plane thawing boundary (one-dimensional case) with the existing exact solution of this problem [86]. The calculation was made for the following conditions:

$$\lambda_1 = 0.168; \quad \lambda_2 = 0.196; \quad \varkappa_1 = 0.231; \quad \varkappa_2 = 0.271;$$

$$\Theta_F = -0.1; \quad \alpha = \infty .$$

During the calculations we restricted ourselves to two approximations. The results of the calculation are as follows: for the approximation method the dimensionless coordinate of the thawing boundary is $S = 0.492\sqrt{t}$; for the exact solution, $S = 0.496\sqrt{t}$. Thus, the accuracy of the method used here proves to be high (about 1% error) in the axisymmetric as well as in the plane one-dimensional cases.

In conclusion to this paragraph, we make the following remark. In work [61] it is confirmed that the radius of thermal influence is proportional to the radius of thawing (freezing). Fig. 4 shows the dependences of the ratio R/S on the dimensionless time t for various values of Θ_F ; these dependences were constructed using the results of the calculation on a computer of system (3.49)-(3.50). As is apparent from this graph, for sufficiently large values of

time, t $>$ 80 to 100 (i.e. 160 to 200 hours for our time scale), the ratio R/S indeed approaches some constant value which depends on Θ_F. Consequently, the confirmation made in work [61] is valid with the stipulation that the proportionality between R and S takes place for sufficiently large values of time and that the coefficient of proportionality depends on Θ_F.

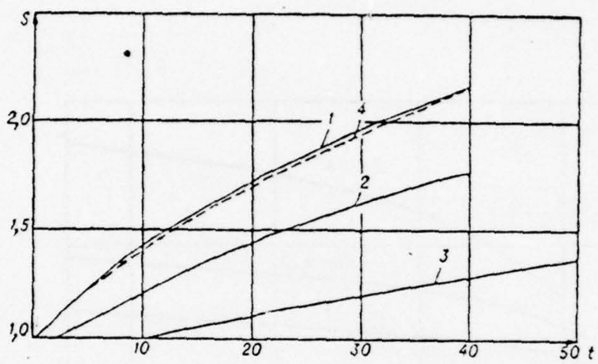

Fig. 3.

1 - $\Theta_F = -0.1$; 2 - $\Theta_F = -0.25$; 3 - $\Theta_F = -0.5$;
4 - $\Theta_F = -0.1$ (exact solution).

The idea of the method of successive approximations proves to be very fruitful when applied to the plane problem of the thawing of the frozen ground around a borehole. As is apparent from the comparisons with the exact solution, the zeroeth approximation (quasistationary solution) gives an accuracy of about 4%, and the first approximation gives an accuracy on the order of 1%. The necessity of the first approximation is due not so much to this increase in accuracy, which may not be very significant for engineering

calculations, as to the fact that it gives the law governing the change in the radius of thermal influence, without which the system of equations describing the motion of the thawing boundary remains open. The solution of the problem in question is obtained either in analytic form for the case of a zero temperature of the frozen ground, or in the form of a system of two ordinary differential equations for the case of a nonzero initial temperature.

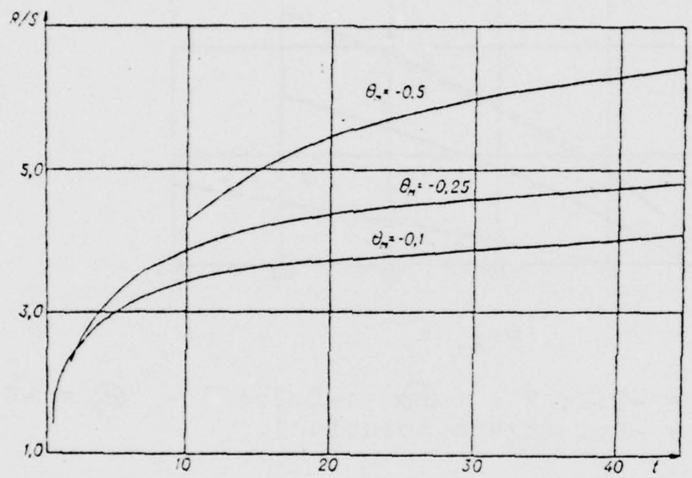

Fig. 4.

The solution of the latter on a computer with a built-in program for integrating ordinary differential equations does not present a problem. The fact that many design and scientific organizations have such computers allows us to recommend the use of the equations derived here for calculating the rate of propagation of the thawing

front around a borehole.

Section 2. Thermal Regime of an Operating Well

Let us consider the problem of determining the temperature distribution in a borehole and the configuration of the thawing front at various moments in time. Let us carry out the solution separately for the cases of oil and gas wells. In contrast to the problem examined in the previous paragraph, here the temperature of the gas (oil) in each section of the borehole is a function of time and of the vertical coordinate of the section. In this case the method developed in the previous chapter is not applicable, since it assumes that the temperature of the gas (oil) in each section is constant.

First, let us solve the problem for frozen ground that is at the thawing temperature or near it. In addition, we shall assume that the temperature of the flow at the entrance to the stratum of frozen ground is constant. In what follows the obtained solutions will be used for the more general case where one cannot neglect the heat flow from the thawing boundary into the frozen region.

Let us rewrite the energy equation (2.83) for gas (oil) in the well in the following form:

$$C_1\left(\frac{\partial T}{\partial t}\right)_z + C_2\left(\frac{\partial T}{\partial z}\right)_t = \frac{\partial \Theta_1}{\partial r}\bigg|_{r=1} - C_3. \tag{3.51}$$

Here the indices next to the derivatives indicate which variable is

held constant during differentiation.

The rate of movement of the thawing boundary is determined from Stefan's condition:

$$-\lambda_1 \frac{\partial \Theta_1}{\partial r}\bigg|_{r=s} = \left(\frac{\partial S}{\partial t}\right)_z. \tag{3.52}$$

The temperature field of the thawed ground is determined, assuming a small rate of movement of the thawing boundary, from the quasistationary heat problem:

$$\frac{1}{r}\cdot\frac{\partial \Theta_1}{\partial r} + \frac{\partial^2 \Theta_1}{\partial r^2} = 0; \tag{3.53}$$

$$\Theta_1|_{r=s} = 0; \quad \frac{\partial \Theta_1}{\partial r}\bigg|_{r=1} = \alpha\,(\Theta_1|_{r=1} - T). \tag{3.54}$$

From this we find:

$$\frac{\partial \Theta_1}{\partial r}\bigg|_{r=1} = -\frac{\alpha T}{\alpha \ln S + 1}; \tag{3.55}$$

$$\frac{\partial \Theta_1}{\partial r}\bigg|_{r=s} = -\frac{\alpha T}{S\,(\alpha \ln S + 1)}, \tag{3.56}$$

Substituting (3.55) into (3.51) and (3.56) into (3.52), we obtain the following system of equations:

$$C_1\left(\frac{\partial T}{\partial t}\right)_z + C_2\left(\frac{\partial T}{\partial z}\right)_t = -\frac{\alpha T}{\alpha \ln S + 1} - C_3. \tag{3.57}$$

$$T\Big|_{z=0} = 1; \tag{3.58}$$

$$\frac{\lambda_1 \alpha T}{S(\alpha \ln S + 1)} = \left(\frac{\partial S}{\partial t}\right)_z; \tag{3.59}$$

$$S\Big|_{t=0} = 1. \tag{3.60}$$

In equation (3.57) let us pass from the variables T, z, t to the variables T, z, S. We have:

$$\left(\frac{\partial T}{\partial t}\right)_z = \left(\frac{\partial T}{\partial S}\right)_z \left(\frac{\partial S}{\partial t}\right)_z;$$
$$\left(\frac{\partial T}{\partial z}\right)_t = \left(\frac{\partial T}{\partial z}\right)_S + \left(\frac{\partial T}{\partial S}\right)_z \left(\frac{\partial S}{\partial z}\right)_t.$$

Rough calculations show that under the conditions of the problem in question

$$\left| \frac{\left(\frac{\partial T}{\partial S}\right)_z \left(\frac{\partial S}{\partial z}\right)_t}{\left(\frac{\partial T}{\partial z}\right)_S} \right| \ll 1.$$

Taking this into account, and also using the expression for $\left(\frac{\partial S}{\partial t}\right)_z$ from (3.59), we obtain equation (3.57) in the following form:

$$C_1 \left(\frac{\partial T}{\partial S}\right)_z \cdot \frac{\lambda_1 \alpha T}{(\alpha \ln S + 1)} + C_2 \left(\frac{\partial T}{\partial z}\right)_S = -\frac{\alpha T}{\alpha \ln S + 1} - C_3; \tag{3.61}$$

$$T\Big|_{z=0} = 1. \tag{3.62}$$

Let us consider the case of an oil well ($C_3 = 0$). The equation of characteristics for equation (3.61) is

$$\frac{S(\alpha \ln S + 1)\,dS}{C_1 \lambda_1 \alpha T} = \frac{dz}{C_2} = \frac{(\alpha \ln S + 1)\,dT}{\alpha T}. \tag{3.63}$$

Hence

$$\frac{S\,dS}{C_1 \lambda_1} = -dT;$$

$$\frac{S^2}{2C_1 \lambda_1} = -T + C.$$

For a characteristic passing through the point T = 1, z = 0, $S = S_0$, we obtain

$$T = 1 + \frac{S_0^2 - S^2}{2C_1\lambda_1}. \qquad (3.64)$$

Let us examine the second relation of equation (3.63):

$$\frac{S(\alpha \ln S + 1)\, dS}{C_1\lambda_1\alpha T} = \frac{dz}{C_2}.$$

Using (3.64), we obtain

$$\frac{S(\alpha \ln S + 1)\, dS}{C_1\lambda_1\alpha\left(1 + \frac{S_0^2 - S^2}{2C_1\lambda_1}\right)} = \frac{dz}{C_2}; \qquad (3.65)$$

$$z\Big|_{S=S_0} = 0. \qquad (3.66)$$

The solution of equations (3.65) and (3.66) gives, in conjunction with (3.64), the desired dependence T(z, s) for a change in S_0 from 1 to ∞. In the practically interesting case

$$|c_1\lambda_1| \ll 1$$

this dependence may be obtained in finite form.

Let us examine the function $z(S, S_0)$. From the boundary

conditions imposed on the characteristic, it follows that

$$z(S, S_0) = 0.$$

Let S_1 be such that

$$z(S_1, S_0) = 1.$$

From (3.65) it follows that

$$\left.\frac{\partial z}{\partial S}\right|_{S \leq S_0} > 0.$$

From this, in view of the fact that

$$z \in [0, 1],$$

it is necessarily true that

$$S \in [S_0, S_1],$$

where $S_1 > S_0$. From an analysis of expression (3.65) it is apparent that $\frac{\partial z}{\partial S}$ monotonically increases over some interval (S_0, S_2) from some finite positive value to $+\infty$. The value of S_2 is determined from the relation

$$\frac{S_0^2 - S_2^2}{2C_1\lambda_1} = -1.$$

Hence

$$S_2^2 - S_0^2 = 2C_1\lambda_1.$$

Since due to the physical meaning of the problem $\frac{\partial z}{\partial S}$ cannot become infinite in the interval $[S_0, S_1]$, $S_2 > S_1$. Taking into account this remark we obtain:

$$S_1^2 - S_0^2 < 2C_1\lambda_1;$$
$$S_1 - S_0 < \frac{2C_1\lambda_1}{S_1 + S_0} < C_1\lambda_1 \ll 1.$$

Consequently,

$$0 \leqslant S - S_0 \leqslant 1 \qquad (3.67)$$

for $S \in [S_0, S_1]$. Integrating (3.65) taking into account (3.67), we obtain

$$z = C_2 \left\{ \frac{S_0}{\sqrt{2C_1\lambda_1 + S_0^2}} \times \right.$$
$$\times \ln \left| \frac{\left(C_1\lambda_1 + S_0^2 + S_0\sqrt{2C_1\lambda_1 + S_0^2}\right)\left(\sqrt{2C_1\lambda_1 + S_0^2} - S\right)}{C_1\lambda_1\left(\sqrt{2C_1\lambda_1 + C_0^2} + S\right)} \right| - \qquad (3.68)$$
$$\left. - \left(1 + \frac{\alpha \ln S_0 + 1}{\alpha}\right) \ln \left| 1 + \frac{S_0^2 - S^2}{2C_1\lambda_1} \right| \right\}.$$

Thus, system of equations (3.64) and (3.68) determines the

characteristic passing through the point T = 1, z = 0, $S = S_0$. Varying the parameter S_0 in these equations from 1 to ∞, we obtain the desired dependence T(z, S), which satisfies equation (3.61) with boundary condition (3.62).

Let us examine the case of a gas well. For gas the effect of the inertia term in the heat flow equation may be neglected [14, 22]. Integrating equations (3.61) and (3.62) taking this remark into account, we obtain

$$T = \frac{1}{\alpha}\left\{[\alpha + C_3(\alpha \ln S + 1)]\exp\left(-\frac{\alpha z}{C_2(\alpha \ln S + 1)}\right) - C_3(\alpha \ln S + 1)\right\}. \tag{3.69}$$

Substituting into (3.59) the function T(z, S) obtained from system (3.64)-(3.68) (for oil) or expression (3.69) (for gas), we find the function t(z, S), which then completes the solution of the problem. In particular, for gas we obtain

$$t = \frac{1}{\lambda_1}\int_1^S \frac{S(\alpha \ln S + 1)\,dS}{[\alpha + C_3(\alpha \ln S + 1)]\exp\left(-\frac{\alpha z}{C_2(\alpha \ln S + 1)}\right) - C_3(\alpha \ln S + 1)}. \tag{3.70}$$

And so, from relations (3.69) and (3.70) one finds the distribution of the temperature of the gas along the borehole as a function of time T(z, t) and the configuration of the thawing boundary at various moments in time S(z, t). The functions are shown in Figures 5 and 6 by solid lines. The calculation was made for the following data: $M = 2.34 \times 10^4$ kg/hour; a = 0.14m; δ_c = 0.03m; δ_w = 0.009m; p_1 = 120atm; λ_c = 0.181 kcal/m-hour-deg; c_p = 0.55 kcal/kg-deg;

$L = 425m$. The remaining data were taken to be the same as in the numerical example of the previous paragraph. The heat transfer coefficient, estimated from criterial dependences, turned out to be equal to $\bar{\alpha} = 370 kcal/m^2$-hour-deg. Under these conditions the dimensionless parameters acquire the following values: $C_2 = 3.14$; $C_3 = 0.310$; $\alpha = 0.55$; $\lambda_1 = 0.168$.

From an analysis of the graph of Fig. 5 it follows that the drop in the temperature of the gas along the borehole proceeds most intensively at the initial moment in time, when the thawing boundary is right against the wall of the borehole. As the thawing boundary withdraws, the heat transfer into the ground, and consequently the rate of the drop in the temperature of the gas along the borehole, decrease. Consequently, from the point of view of the possibility of hydrate formation in wells in frozen ground, the initial period of exploitation of the well is the most hazardous.

The graph of Fig. 6 illustrates the obvious fact that the rate of thawing of frozen ground around the borehole decreases as daytime is approached, in connection with the drop in the temperature of the gas.

In order to appraise the accuracy of the presented method, a comparison was carried out of the results obtained using this method with the exact solution of problem (2.72)-(2.84), obtained with the aid of numerical methods on a computer. The exact solution is shown in Figures 5 and 6 by dotted lines. Comparison of the approximate and exact solutions shows that the proposed method is highly accurate.

For example, for t = 10 the dimensionless mouth temperature of the gas for the exact solution is 0.7765, and for the approximate solution is 0.7735, i.e. the error is about 0.45%. For large values of t the error is even smaller. The error in determining the coordinate of the thawing front is about 5%.

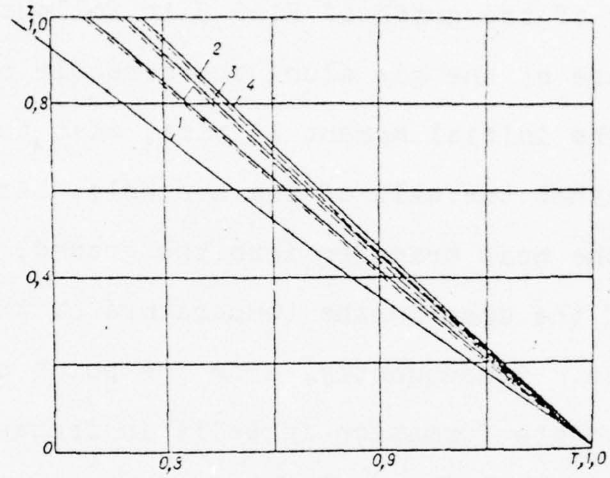

Fig. 5.

——————— approximate solution;

--------- exact solution.

1 - t = 0; 2 - t = 10; 3 - t = 30; 4 - t = 50.

Let us apply the derived solution to the case of a nonuniform frozen stratum with a zero initial temperature. Let the stratum consist of a series of horizontal layers with different thermophysical properties, which are constant in each layer. The dimensionless parameters for the ith layer we shall denote by C_{2i}, C_{3i}, α_i, and $\lambda_1^{(i)}$. Let us denote the coordinate of the interface between the

ith and (i + 1)th layer by l_i' (Fig. 7). Examining the case of a gas well, within the limits of the ith layer the heat flow equation is written in the form:

$$C_{2i}\left(\frac{\partial T}{\partial z}\right)_S = -\frac{\alpha_i T}{\alpha_i \ln S + 1} - C_{3i} \quad (3.71)$$

for $l_{i-1} < z < l_i$,

$$T\big|_{z=l_{i-1}} = T_{0i}(t). \quad (3.72)$$

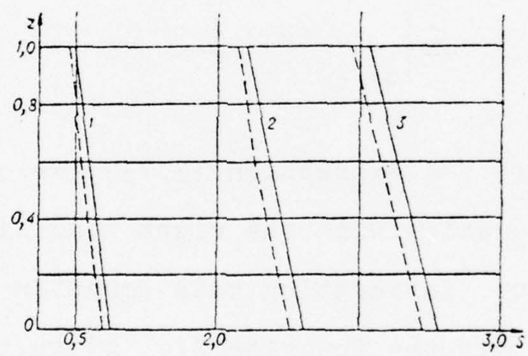

Fig. 6.

____________ approximate solution;

----------- exact solution.
1 - t = 10; 2 - t = 30; 3 - t = 50.

Here $T_{0i}(t)$ is the temperature of the gas at the entrance to the ith layer. The solution of problem (3.71) and (3.72) is as follows:

$$T = \frac{1}{\alpha_i}\left\{[\alpha_i T_{0i}(t) + C_{3i}(\alpha_i \ln S + 1)] \exp\left(-\frac{\alpha_i (z - l_{i-1})}{C_{2i}(\alpha_i \ln S + 1)}\right) - C_{3i}(\alpha_i \ln S + 1)\right\}. \qquad (3.73)$$

The function S(z, t) is determined from equation (3.59), which for the ith layer is written in the form:

$$\left(\frac{\partial S}{\partial t}\right)_z = \frac{\lambda_1^{(i)} \alpha_i T}{S(\alpha_i \ln S + 1)}; \qquad (3.74)$$

$$S\Big|_{t=0} = 1. \qquad (3.75)$$

Substituting into (3.74) expression (3.73), we obtain an ordinary differential equation with the right hand side expressed explicity in S, z, and t. Integrating this equation numerically for each fixed z, we obtain the function S(z, t) in the ith layer. Substituting this function into (3.73), we obtain T(z, t). Recursion relations for the temperature of the gas at the entrance to each layer $T_{0i}(t)$ are easily obtained from (3.73), assuming $z = l_i$:

$$T_{0(i+1)}(t) = T\big|_{z=l_i} = \frac{1}{\alpha_i}\left\{[\alpha_i T_{0i}(t) + C_{3i}(\alpha_i \ln S + 1)] \exp\left(-\frac{\alpha_i (l_i - l_{i-1})}{C_{2i}(\alpha_i \ln S + 1)}\right) - C_{3i}(\alpha_i \ln S + 1)\right\}. \qquad (3.76)$$

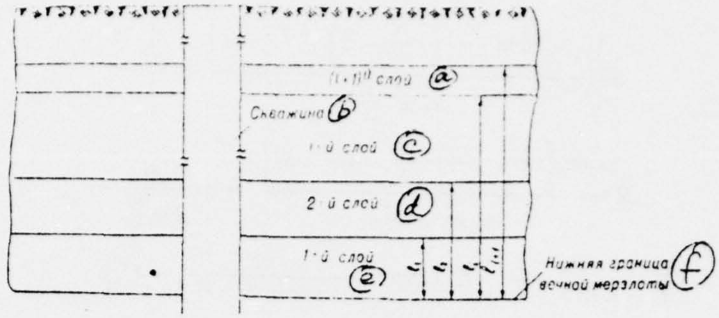

Fig. 7.

Key: a - (i + 1)th layer
b - borehole
c - ith layer
d - 2nd layer
e - ist layer
f - lower boundary of permafrost

Thus, solving successively problem (3.71)-(3.74) for each layer, one may determine the distribution of the temperature of the gas and the configuration of the thawing front at various moments in time for the entire nonuniform stratum.

Let us now examine the case where the initial temperature of the frozen ground is different from zero. Taking into account expressions (3.56) and (3.69), we obtain for a gas well

$$\left.\frac{\partial\Theta_1}{\partial r}\right|_{r=S} = -\frac{\alpha T}{S(\alpha \ln S + 1)} = -\frac{1}{S}\left\{\left[\frac{\alpha}{\alpha \ln S + 1} + C_3\right]\cdot \exp\left(-\frac{l\alpha z}{C_2(\alpha \ln S + 1)}\right) - C_3\right\}.$$

In order to determine the heat flow in the frozen region let us use expression (3.30); then Stefan's condition (2.81) acquires

the form

$$\dot S\left\{\frac{\lambda_2\Theta_u\left[2\ln\frac{R}{S}\left(\ln\frac{S}{R}-1\right)+\frac{R^2}{S^2}-1\right]}{4\varkappa_2\ln^3\frac{R}{S}}-1\right\}+\dot R\,\frac{\lambda_2\Theta_u\left[(R^2+S^2)\ln\frac{R}{S}-R^2+S^2\right]}{4\varkappa_2 RS\ln^3\frac{R}{S}}=-\frac{\lambda_2\Theta_u}{S\ln\frac{R}{S}}-\frac{\lambda_1}{S}\left\{\left(\frac{\alpha}{\alpha\ln S+1}+C_3\right)\exp\left(-\frac{\alpha z}{C_2(\alpha\ln S+1)}\right)-C_3\right\}. \qquad (3.77)$$

$\Theta_M \equiv \Theta_F$

Adjoining to this equation (3.32), we obtain a system of two ordinary differential equations of the first order with boundary conditions (3.33) and (3.34). The variable z appears here as a parameter. Thus, this system may be integrated for each value of z, i.e. one may determine the function S(z, t). Substituting this relation into expression (3.69), we obtain T(z, t). System (3.77) and (3.32) has been integrated on a "Nairi" computer using a standard program for integration using the Runge-Kutta method. In order to do this, equations (3.77) and (3.32) were solved for $\dot S$ and $\dot R$. The following was finally obtained:

$$\dot S=\left\{\lambda_2\Theta_u\left(2\ln\frac{R}{S}-1+\frac{S^2}{R^2}\right)-\lambda_1\left[\left(\frac{\alpha}{\alpha\ln S+1}+C_3\right)\cdot\exp\left(-\frac{\alpha z}{C_2(\alpha\ln S+1)}\right)-C_3\right]\left[2\ln\frac{R}{S}\left(\ln\frac{S}{R}+1\right)-1+\frac{S^2}{R^2}\right]\right\}\Bigg/ S\left\{\frac{\lambda_2\Theta_u\left(4\ln\frac{R}{S}+2-\frac{S^2}{R^2}-\frac{R^2}{S^2}\right)}{4\varkappa_2\ln\frac{R}{S}}-2\ln\frac{R}{S}\left(\ln\frac{S}{R}+1\right)+1-\frac{S^2}{R^2}\right\}; \qquad (3.78)$$

$\Theta_M \equiv \Theta_F$

$$
\begin{aligned}
\dot{R} = \Bigg\{ & \frac{\lambda_2 \Theta_M \left(\frac{R^2}{S^2} - 2 \ln \frac{R}{S} - 1 \right)}{R} - \lambda_1 \Bigg[\left(\frac{\alpha}{\alpha \ln S + 1} + C_3 \right) \times \\
& \times \exp\left(- \frac{\alpha z}{c_2 (\alpha \ln S + 1)} \right) - C_3 \Bigg] \frac{R^2 - S^2 - (R^2 + S^2) \ln \frac{R}{S}}{RS^2} - \\
& - \frac{4 \varkappa_2 \ln^2 \frac{R}{S}}{R} \Bigg\} \Bigg/ \Bigg\{ 2 \ln \frac{R}{S} \left(\ln \frac{S}{R} - 1 \right) - 1 + \frac{S^2}{R^2} - \\
& - \frac{\lambda_2 \Theta_M}{4 \varkappa_2 \ln \frac{R}{S}} \left[4 \ln^2 \frac{R}{S} + 2 - \frac{S^2}{R^2} - \frac{R^2}{S^2} \right] \Bigg\}.
\end{aligned} \tag{3.79}
$$

$\Theta_M = \Theta_F$

The boundary conditions are as follows:

$$
S_{t=t_m} = 1; \qquad R_{t=t_m} = R_m. \tag{3.80}
$$

System (3.78)-(3.79) was integrated for the following values of the dimensionless parameters: $\lambda_1 = 0.168$, $\lambda_2 = 0.196$, $\varkappa_2 = 0.271$, $\alpha = 0.55$, $c_2 = 3.14$, $\Theta_F = -0.10$, $\alpha_2 = 0.472$, $C_3 = 0.31$.

The results of the calculations are shown in Figures 8 and 9, in which the solid lines show the graphs of the functions S(z, t) and T(z, t), and the dotted lines show the results of the exact solution of problem (2.72)-(2.84) for the same values of the dimensionless parameters, obtained by reduction to a system of integral equations. As in the case of a zero initial temperature, one may note the good coincidence of the exact and approximate solutions. The maximum error in determining the temperature of the gas does not exceed 1%, and the maximum error in determining the coordinate of the thawing front does not exceed 5%.

In conclusion we note that the described methodology is also applicable for thermal calculations for boreholes crossing frozen as well as unfrozen strata. Let an unfrozen stratum of thickness L_1 be situated between a frozen area and a producing bed.

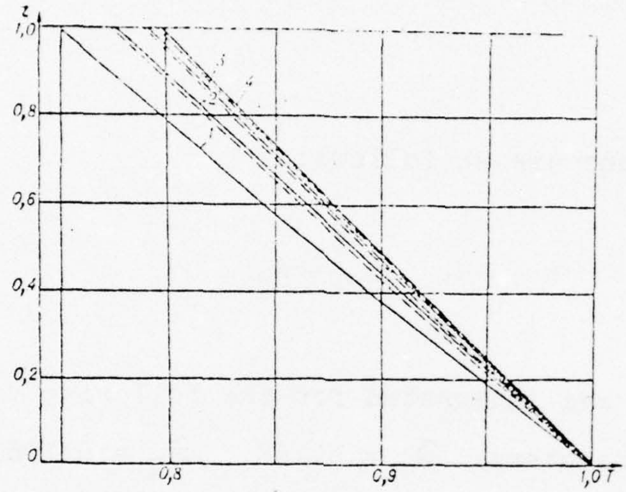

Fig. 8.

________ approximate solution;

--------- exact solution.

1 - t = 0; 2 - t = 10; 3 - t = 30; 4 - t = 50

The thermal regime of a gas well in a section of an unfrozen stratum is given by the formula of E. B. Chekalyuk [19]. If one calls the temperature of the producing bed T_{bed}, then the law governing the change in the temperature at the entrance to the stratum of permafrost will have the form [next page]:

$$T_{вх}(t) = \Theta_н - C_3 \ln(1 + \sqrt{\varkappa t}) + (1 - \Theta_н + C_3 \ln(1 + \sqrt{\varkappa t})) \times \exp\left(-\frac{L_1}{C_2 \ln(1 + \sqrt{\varkappa t})}\right).$$

$T_{Bx} \equiv T_{entrance}$

$\Theta_H \equiv \Theta_u$

Here

$$x = \frac{\pi \bar{\lambda}_1 t_0}{C_r a^2};$$

C_r, $\bar{\lambda}_1$ are respectively the specific heat and coefficient of thermal conductivity of the unfrozen ground, and Θ_u is the temperature of the unfrozen ground.

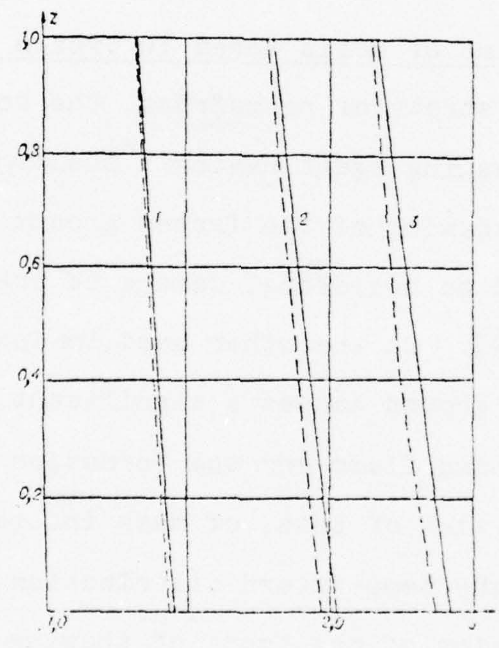

Fig. 9.

_____ approximate solution; ----- exact solution;
1 - t = 10; 2 - t = 30; 3 - t = 50.

Integrating for the case of a gas well equation (3.61) with the boundary condition

$$T\Big|_{z=0} = T_0(t),$$

we obtain

$$T_r = \frac{1}{\alpha}\Big\{[\alpha T_{rx}(t) + C_3(\alpha \ln S + 1)] \times \exp\left(-\frac{\alpha z}{C_2(\alpha \ln S + 1)}\right) - C_3(\alpha \ln S + 1)\Big\}. \qquad (3.81)$$

$T_r \equiv T_g$

$T_{Bx} \equiv T_{entrance}$

All subsequent calculations are carried out just as was described above, replacing (3.69) by (3.81).

Section 3. Thermal Regime of Wells Bored in Frozen Rock

In the presence of strata of permafrost, the boring and flushing of boreholes using a flushing agent having a positive temperature are accompanied by the thawing of the frozen ground around the borehole, which may lead to accidental damage to the cohesiveness of the thawed ground [84]. On the other hand, the low temperature of the strata of frozen ground causes a significant lowering of the temperature of the flushing fluid and the formation of ice plugs in the drill pipes. In view of this, of much interest is the problem of determining the temperature distribution in the flushing fluid and the configuration of the front of thawing of the permafrost.

In the present paragraph we propose an approximation method for solving this problem for the case of the flushing of a borehole bored

in frozen ground. The corresponding problem for ordinary (unfrozen) ground was solved by I. A. Charnyy [9]. Here his solution, using the method developed in Section 2, is applied to frozen ground. As in Section 2, let us first give the solution for frozen ground with an initial zero temperature, and then let us apply this solution to the case of an arbitrary initial temperature.

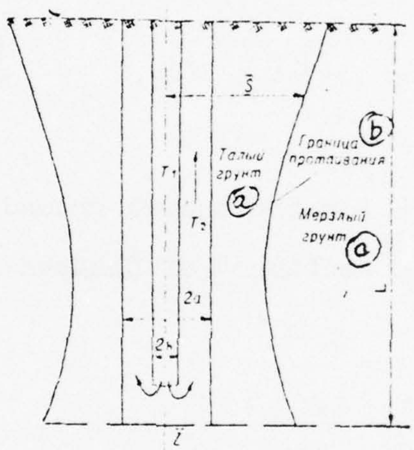

Fig. 10.

Key: a - thawed ground
b - thawing boundary
c - frozen ground

Let us examine the process of the flushing of a well (Fig. 10) bored in uniform frozen ground with an initial temperature of 0°C. After [9], the heat flow equation for the flushing fluid in the central pipe and in the annular space is written as follows [next page]:

$$cG\left(\frac{\partial \bar{T}_2}{\partial \bar{z}}\right)_{\bar{t}} = 2\bar{\lambda}_1 \frac{\partial \bar{\Theta}_1}{\partial \bar{r}}\bigg|_{\bar{r}=a} \pi a - k_{1-2} \cdot 2\pi b (\bar{T}_1 - \bar{T}_2); \tag{3.82}$$

$$cG\left(\frac{\partial \bar{T}_1}{\partial \bar{z}}\right)_{\bar{t}} = -k_{1-2} 2\pi b (\bar{T}_1 - \bar{T}_2). \tag{3.83}$$

The boundary conditions are as follows:

$$\bar{T}_1|_{\bar{z}=0} = T_{BX}; \qquad \bar{T}_1|_{\bar{z}=L} = \bar{T}_2|_{\bar{z}=L}. \tag{3.84}$$

$T_{Bx} \equiv T_{entrance}$

The heat conduction equation for the thawed ground, in accordance with the assumptions made in Section 4 of Chapter 1, is written as follows:

$$\frac{\partial \bar{\Theta}_1}{\partial \bar{t}} = a_1\left(\frac{1}{\bar{r}}\frac{\partial \bar{\Theta}_1}{\partial \bar{r}} + \frac{\partial^2 \bar{\Theta}_1}{\partial \bar{r}^2}\right) \quad a < \bar{r} < \bar{S}(\bar{t}, \bar{z}) \tag{3.85}$$

The boundary conditions are as follows:

$$\bar{\Theta}_1|_{\bar{r}=\bar{S}} = 0; \tag{3.86}$$

$$\bar{\lambda}_1 \frac{\partial \bar{\Theta}_1}{\partial \bar{r}}\bigg|_{\bar{r}=a} = \bar{\alpha}(\bar{\Theta}_1|_{\bar{r}=a} - \bar{T}_2). \tag{3.87}$$

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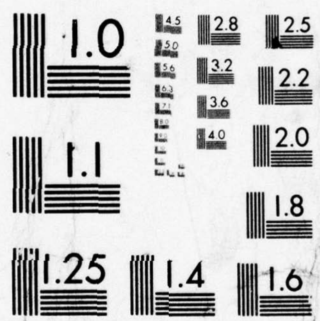

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The rate of advancement of the ground thawing boundary is determined from Stefan's condition:

$$l\rho_1 \left(\frac{\partial \bar{S}}{\partial \bar{t}} \right)_{\bar{z}} = -\bar{\lambda}_1 \frac{\partial \bar{\Theta}_1}{\partial \bar{r}} \Big|_{\bar{r}=\bar{S}}. \qquad (3.88)$$

Let us introduce the dimensionless quantities

$$T_i = \frac{\bar{T}_i}{T_{\text{вх}}}; \quad z = \frac{\bar{z}}{L}; \quad r = \frac{\bar{r}}{a}; \quad \Theta_1 = \frac{\bar{\Theta}_1}{T_{\text{вх}}}; \quad S = \frac{\bar{S}}{a}; \quad \alpha = \frac{a\bar{\alpha}}{\bar{\lambda}_1};$$

$T_{Bx} \equiv T_{entrance}$

$$\alpha_1 = \frac{2\pi k_1 \ldots bL}{cG}; \quad \alpha_2 = \frac{2\pi\bar{\alpha}aL}{cG}; \quad \lambda_1 = \frac{\bar{\lambda}_1 \cdot T_{\text{вх}} \cdot t_0}{\rho_1 l a^2}; \quad \varkappa_1 = \frac{a_1 t_0}{a^2}.$$

In these variables equations (3.82)-(3.88) acquire the form:

$$\left(\frac{\partial T_2}{\partial z} \right)_t = -\frac{2\pi L \bar{\lambda}_1}{cG} \frac{\partial \Theta_1}{\partial r} \Big|_{r=1} - \alpha_1 (T_1 - T_2); \qquad (3.89)$$

$$\left(\frac{\partial T_1}{\partial z} \right)_t = -\alpha_1 (T_1 - T_2); \qquad (3.90)$$

$$\frac{\partial \Theta_1}{\partial t} = \varkappa_1 \left(\frac{1}{r} \frac{\partial \Theta_1}{\partial r} + \frac{\partial^2 \Theta_1}{\partial r^2} \right); \qquad (3.91)$$

$$\left(\frac{\partial S}{\partial t}\right)_z = -\lambda_1 \frac{\partial \Theta_1}{\partial r^2}\bigg|_{r=S}; \tag{3.92}$$

$$T_1|_{t=0} = 1; \quad T_1|_{z=1} = T_2|_{z=1}; \quad S|_{t=0} = 1;$$

$$\Theta_1|_{r=S} = 0; \quad \frac{\partial \Theta_1}{\partial r}\bigg|_{r=1} = \alpha\,(\Theta_1|_{r=1} - T_2). \tag{3.93}$$

In view of the fact that the rate of advancement of the ground thawing boundary is small, one may consider that the distribution of the temperatures in the thawed region differs only slightly from the stationary distribution. Taking into account this remark we obtain

$$\frac{\partial \Theta_1}{\partial r}\bigg|_{r=S} = -\frac{\alpha T_2}{S\,(\alpha \ln S + 1)}; \quad \frac{\partial \Theta_1}{\partial r}\bigg|_{r=1} = -\frac{\alpha T_2}{\alpha \ln S + 1}. \tag{3.94}$$

Substituting relations (3.94) into (3.89), (3.90), and (3.92), we obtain:

$$\left(\frac{\partial T_2}{\partial z}\right)_t = -\frac{\alpha_2 T_2}{\alpha \ln S + 1} + \alpha_1 (T_1 - T_2); \tag{3.95}$$

$$\left(\frac{\partial T_1}{\partial z}\right)_t = -\alpha_1 (T_1 - T_2); \tag{3.96}$$

$$\left(\frac{\partial S}{\partial t}\right)_z = \frac{\lambda_1 \alpha T_2}{S(\alpha \ln S + 1)}. \qquad (3.97)$$

Thus, the problem reduces to the solution of a system of the three differential equations (3.95)-(3.97) with boundary and initial conditions (3.93).

In system (3.95)-(3.96) let us pass from the variables (T_1, T_2, z, t) to the variables (T_1, T_2, z, S), taking into account that

$$\left(\frac{\partial T_i}{\partial z}\right)_t = \left(\frac{\partial T_i}{\partial z}\right)_S + \left(\frac{\partial T_i}{\partial S}\right)_z \left(\frac{\partial S}{\partial z}\right)_t.$$

Rough calculations show that for boreholes having a sufficiently large diameter ($\geqslant$ 190mm), $\left(\frac{\partial T_i}{\partial S}\right)_z \left(\frac{\partial S}{\partial z}\right)_t$ is small. Taking this into account, equations (3.95) and (3.96) may be rewritten in the form:

$$\left(\frac{\partial T_2}{\partial z}\right)_S = \frac{\alpha_2 T_2}{\alpha \ln S + 1} - \alpha_1 (T_1 - T_2); \qquad (3.98)$$

$$\left(\frac{\partial T_1}{\partial z}\right)_S = -\alpha_1 (T_1 - T_2). \qquad (3.99)$$

Eliminating from this system T_2, we obtain

$$\left(\frac{\partial^2 T_1}{\partial z^2}\right)_S - \frac{\alpha_2}{\alpha \ln S + 1}\left(\frac{\partial T_1}{\partial z}\right)_S - \frac{\alpha_2 \alpha_1 T_1}{\alpha \ln S + 1} = 0. \qquad (3.100)$$

The boundary conditions are as follows:

$$T_1\Big|_{z=0} = 1; \qquad (3.101)$$

$$\left.\frac{\partial T_1}{\partial z}\right|_{z=1} = 0. \qquad (3.102)$$

The latter condition is obtained at once from equation (3.99) and boundary conditions (3.93).

Solving for each fixed S the boundary-value problem (3.100)-(3.102), we obtain

$$T_1 = \frac{r_2 e^{r_2 + r_1 z} - r_1 e^{r_1 + r_2 z}}{r_2 e^{r_2} - r_1 e^{r_1}}, \qquad (3.103)$$

where

$$r_{1,2} = \frac{\alpha_2}{2(\alpha \ln S + 1)} \pm \sqrt{\frac{\alpha_2^2}{4(\alpha \ln S + 1)^2} + \frac{\alpha_2 \alpha_1}{\alpha \ln S + 1}}.$$

Analogously, for T_2 we obtain the equation

$$\left(\frac{\partial^2 T_2}{\partial z^2}\right)_S - \frac{\alpha_2}{\alpha \ln S + 1}\left(\frac{\partial T_2}{\partial z}\right)_S - \frac{\alpha_2 \alpha_1 T_2}{\alpha \ln S + 1} = 0. \tag{3.104}$$

The boundary conditions are as follows:

$$T_2|_{z=1} = T_1|_{z=1}; \tag{3.105}$$

$$\left.\frac{\partial T_2}{\partial z}\right|_{z=1} = \left.\frac{\alpha_2 T_1}{\alpha \ln S + 1}\right|_{z=1}. \tag{3.106}$$

The solution of problem (3.104)-(3.106) for each fixed S has the form

$$T_2 = \frac{r_2 e^{r_1 + r_2 z} - r_1 e^{r_2 + r_1 z}}{r_2 e^{r_2} - r_1 e^{r_1}}. \tag{3.107}$$

Thus, we have obtained $T_2(z, S)$. Substituting (3.107) into (3.97), we obtain an ordinary differential equation, the right hand side of which is a known function of z and S. Solving this equation numerically for each fixed z, we obtain the function S(z, t). Substituting this expression into (3.93) and (3.107), we obtain $T_1(z, t)$ and $T_2(z, t)$.

Let us examine the case of the flushing of a borehole that is bored in frozen ground with an initial temperature different from

zero. Taking into account expressions (3.94) and (3.107) we obtain for the thawed region

$$\left.\frac{\partial \Theta_1}{\partial r}\right|_{r=S} = -\frac{\alpha T_2}{S(\alpha \ln S + 1)} = -\frac{\alpha}{S(\alpha \ln S + 1)} \cdot \frac{r_2 e^{r_1 + r_2 z} - r_1 e^{r_2 + r_1 z}}{r_2 e^{r_2} - r_1 e^{r_1}}. \qquad (3.108)$$

In order to find the heat flow into the frozen region let us use expression (3.30). Then Stefan's condition (2.81) acquires the form

$\Theta_M = \Theta_F$

$$\dot{S}\left\{\frac{\lambda_2 \Theta_M \left[2 \ln \frac{R}{S} \ln \frac{S}{Re} + \frac{R^2}{S^2} - 1\right]}{4\varkappa_2 \ln^3 \frac{R}{S}} - 1\right\} +$$
$$+ \dot{R}\,\frac{\lambda_2 \Theta_M \left[(R^2 + S^2) \ln \frac{R}{S} - R^2 + S^2\right]}{4\varkappa_2 R S \ln^3 \frac{R}{S}} = -\frac{\lambda_2 \Theta_M}{S \ln \frac{R}{S}} -$$
$$- \frac{\lambda_1 \alpha}{S(\alpha \ln S + 1)} \cdot \frac{r_2 e^{r_1 + r_2 z} - r_1 e^{r_2 + r_1 z}}{r_2 e^{r_2} - r_1 e^{r_1}}. \qquad (3.109)$$

Adding here equation (3.32), we obtain a system of two ordinary differential equations relative to the functions S(z, t) and R(z, t). As initial conditions we may use conditions (3.33) and (3.34). Solving this system for each fixed z, we obtain the desired function S(z, t), and, substituting this function into expressions (3.103) and (3.107), we obtain $T_1(z, t)$ and $T_2(z, t)$, which then completes the solution of the problem.

Let us calculate the temperature of the flushing fluid during the flushing of a well in frozen ground with a zero initial temperature. The calculation will be made for the following beginning

data: $T_{entrance} = 20°C$; $L = 440m$; $a = 0.095m$; $b = 0.052m$; $\nu_\ell = 0.8 \times 10^{-6} m^2/sec$; $\lambda_\ell = 0.147 \times 10^{-3}$ kcal/m-sec-deg; $k_{1-2} = 0.396$ kcal/m^2-sec-deg; $\overline{\alpha} = 0.760$ kcal/m^2-sec-deg; $\overline{\lambda}_1 = 0.428 \times 10^{-3}$ kcal/m-deg-sec; $c = 1$ kcal/dg-deg; $t_0 = 2$ hours; $G = 18$kg/sec.

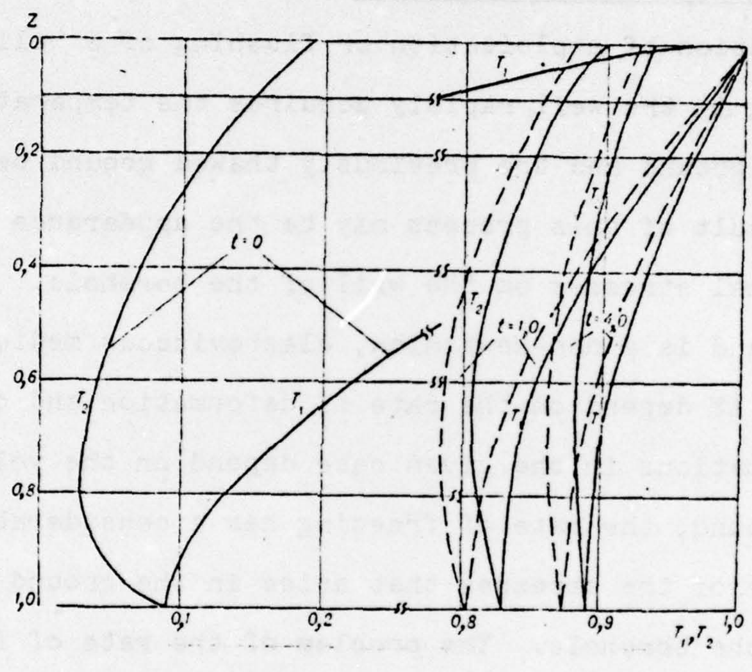

Fig. 11.

The results of the calculation are presented in Figures 11 and 12. The solid lines show the distributions of the temperatures T_1 and T_2, and also the configuration of the thawing front S, calculated for various moments in time. The dotted lines show the results of the exact solution of problem (3.89)-(3.93), obtained using numerical methods. Comparison of the approximate and exact solutions

shows that the applied method is highly accurate. Thus, for t = 4 the temperature of the flushing fluid at the exit from the annular space for the exact solution is 0.919, and for the approximate solution the value is 0.931. Consequently, the error in determining the coordinate of the thawing front is about 1.2%.

Section 4. Calculation of the Rate of Freezing of Previously Thawed Ground After Shutdown of a Well

After cessation of exploitation or flushing of a well, the liquid remaining in the well rapidly acquires the temperature of the surrounding ground and the previously thawed ground begins to freeze. The result of this process may be the appearance of significant normal stresses on the wall of the borehole.

Frozen ground is a non-Newtonian, elastoviscous medium, i.e. the stresses in it depend on the rate of deformation and on time. Since the deformations in the given case depend on the volume of the freezing ground, the rate of freezing has a considerable effect on the magnitude of the stresses that arise in the ground and on the wall of the borehole. The problem of the rate of freezing of the ground around a borehole thus acquires great practical importance.

The temperature field in the thawed ground is described by equation (2.72) with boundary conditions (2.73) and $\left.\frac{\partial \theta_1}{\partial r}\right|_{r=1} = 0$, i.e. we assume that the temperature of the liquid rapidly drops down to the temperature of the surrounding ground, in connection with which the heat flow at the wall of the borehole will be equal to zero.

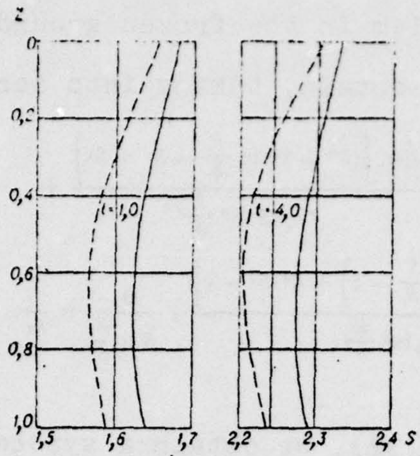

Fig. 12.

The temperature field in the frozen ground is described by equation (2.78) with boundary condition (2.80) and initial condition (2.79), where by t_m will be understood the time at which freezing begins, and by $\Theta_c(r,z)$ will be understood the temperature field in the frozen ground at this time. Here it is necessary to add a condition at the freezing boundary [(2.81)].

As rough calculations show, the temperature of the thawed ground after a very short time interval (in comparison with the total freezing time) becomes close to 0°C, i.e. during practically all of the freezing time the flow from the thawed region is small and may be neglected when compared with the heat flow in the direction of the frozen ground. An analogous assumption is made in work [54]. Thus, the condition at the freezing boundary may be written as

$$\lambda_2 \frac{\partial \Theta_2}{\partial r}\bigg|_{r=S} = \frac{\partial S}{\partial t}. \tag{3.110}$$

Solving the heat problem in the frozen ground using the method presented in Section 1, we obtain, taking into account (3.110),

$$\left.\frac{\partial\Theta_2}{\partial r}\right|_{r=S} = \frac{\dot{R}\Theta_M\left[(R^2+S^2)\ln\frac{R}{S} - R^2 + S^2\right]}{4\varkappa_2 RS\ln^3\frac{R}{S}} +$$

$\Theta_M \equiv \Theta_F$

$$+\frac{\dot{S}\Theta_M\left[2\ln\frac{R}{S}\left(\ln\frac{S}{R}-1\right)+R^2/S^2-1\right]}{4\varkappa_2\ln^3\frac{R}{S}} + \frac{\Theta_M}{S\ln\frac{R}{S}} = \frac{\dot{S}}{\lambda_2}. \tag{3.111}$$

Adding to this equation (3.32), we obtain a system of two ordinary differential equations of the first order relative to the functions S(t) and R(t). Solving this system for $\dot{S}$ and $\dot{R}$, we finally obtain:

$$\dot{S} = \frac{\lambda_2\Theta_M}{S}\left(2\ln\frac{R}{S}+\frac{S^2}{R^2}-1\right)\Bigg/\left\{\frac{\lambda_2\Theta_M\left(4\ln^2\frac{R}{S}+2-\frac{R^2}{S^2}-\frac{S^2}{R^2}\right)}{4\varkappa_2\ln\frac{R}{S}} -\right.$$

$\Theta_M \equiv \Theta_F$

$$\left.-\frac{S^2}{R^2}+1-2\ln\frac{R}{S}\left(1-\ln\frac{R}{S}\right)\right\}; \tag{3.112}$$

$$\dot{R} = \left\{-\frac{\lambda_2\Theta_M\left(R^2-S^2-2S^2\ln\frac{R}{S}\right)}{RS^2}+\frac{4\varkappa_2\ln^2\frac{R}{S}}{R}\right\}\Bigg/$$

$$\Bigg/\left\{\frac{\lambda_2\Theta_M}{4\varkappa_2\ln\frac{R}{S}}\left(4\ln^2\frac{R}{S}+2-\frac{R^2}{S^2}-\frac{S^2}{R^2}\right)-\frac{S^2}{R^2}+1-\right.$$

$$\left.-2\ln\frac{R}{S}\left(1-\ln\frac{R}{S}\right)\right\}; \tag{3.113}$$

$$R|_{t=t_m} = R_m;\quad S|_{t=t_m} = S_m. \tag{3.114}$$

Here the values for R_m and S_m are obtained from the solution of the problem of thawing.

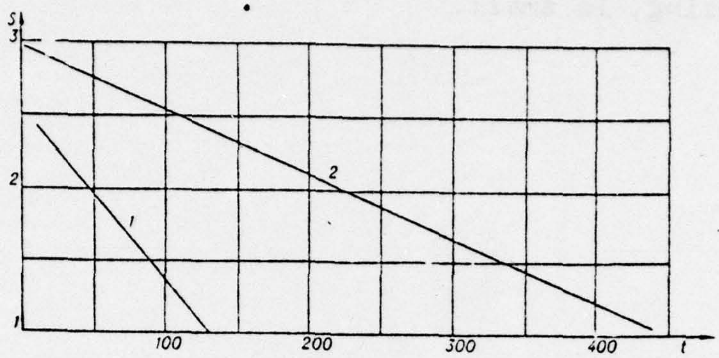

Fig. 13.

1 - $\Theta_F = -0.25$; 2 - $\Theta_F = -0.1$.

System (3.112)-(3.113) was integrated using a standard program on a "Nairi" computer for the values of the dimensionless parameters cited above. The results of the calculation are shown in Fig. 13. From these results one may conclude that the freezing of ground that had thawed to some radius S_m requires much more time than the thawing to this radius. For example, for $\Theta_F = -0.1$ and $S_m = 3$ the thawing time is 168 hours, and the refreezing time is 900 hours. This may easily be explained from the following considerations. The flushing and exploitation of wells is an intensive thermal process, connected with the effect of large amounts of warm fluid on frozen ground. In this connection, the heat flow at the wall of the borehole is great, and the rate of thawing is sufficiently great.

At the same time, the process of refreezing is much less intensive, since the temperature of the frozen ground usually differs little from 0°C, and the heat flow toward the frozen ground, which determines the rate of freezing, is small.

APPENDIX 1

1. Reduction of the Problem of the Heat Exchange Between a Well and Frozen Ground to a System of Integral Equations

In order to appraise the accuracy of the above-developed approximation methods for calculating the thermal regime of wells in a region of permafrost, it is necessary to compare the results obtained using these methods with the exact solution of problem (2.72)-(2.84). In order to obtain the latter let us use the method of reducing the axisymmetric Stefan problem to a system of integral equations [21]. Work [21] examines the one-dimensional axisymmetric problem of the melting of a cylinder of infinite length, the temperature of the surface of which varies according to an arbitrary law. In our case we are concerned with two-dimensional problems in a semi-infinite region with a cylindrical hole, in which the temperature varies with time and with height. However, using the methodology developed in [21] and [86], it proves to be possible to reduce this problem to a system of integral and ordinary differential equations.

Let us examine the region EABF (Fig. 14), bounded by the characteristics AB and EF and the curves AE: $r = r_1(t)$ and BF: $r = r_2(t)$. Let us consider sufficiently smooth functions $\varphi(r,t)$ such that

$$\mathfrak{L}(\varphi) \equiv a^2\left(\frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r}\right) - \frac{\partial\varphi}{\partial t} = 0;$$

$$\mathfrak{M}(\psi) \equiv a^2\left(\frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r}\right) + \frac{\partial\psi}{\partial t} = 0.$$

According to Green's formula, for the region PABQ we have

$$\int_{PABQP} \left[r\varphi\psi dr + a^2 r \left(\psi \frac{\partial\varphi}{\partial r} - \varphi \frac{\partial\psi}{\partial r} \right) dt \right] =$$

$$= \iint_S \left[a^2 r\psi \left(\frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r} \frac{\partial\varphi}{\partial r} \right) - a^2 r\varphi \left(\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r} \frac{\partial\psi}{\partial r} \right) - r\varphi \frac{\partial\psi}{\partial t} - \right.$$

$$\left. - r\psi \frac{\partial\varphi}{\partial t} \right] dr dt = \iint_S [r\psi \mathfrak{L}(\varphi) - r\varphi \mathfrak{M}(\psi)] dr dt = 0.$$

Hence

$$\int_{PQ} r\varphi\psi dr = \int_{AB} r\varphi\psi dr + \int_{BQ+PA} \left[r\varphi\psi dr + a^2 r \left(\psi \frac{\partial\varphi}{\partial r} - \varphi \frac{\partial\psi}{\partial r} \right) dt \right]. \quad \text{(A.1)}$$

Let $\varphi(r, t) = \Theta(r, t)$ be some solution of the heat conduction equation. Let us examine the function of an instantaneous cylindrical source of heat

$$E_0(r, \xi, \varkappa(t - \tau)) = \frac{\exp\left(-\frac{r^2 + \xi^2}{4\varkappa(t - \tau)} \right)}{2\varkappa(t - \tau)} I_0\left(\frac{r\xi}{2\varkappa(t - \tau)} \right).$$

For this function we have

$\mathfrak{L}(E_0) = 0$ over r and t;

$\mathfrak{M}(E_0) = 0$ over ξ and τ.

Let us examine the region PABQ in the coordinates ξ, τ and the points M(r, t) and M_1(r, t + h). Let us assume that

$$\varphi = \Theta(\xi, \tau);\ \psi = E_0(r, \xi, \varkappa(t + h - \tau)).$$

Then formula (A.1) acquires the form

$$\int_{PQ} \xi\Theta(\xi, t) \frac{\exp\left(-\frac{r^2+\xi^2}{4\varkappa h}\right) I_0\left(\frac{r\xi}{2\varkappa h}\right)}{2\varkappa h} d\xi = \int_{AB} \xi\Theta(\xi, \tau) \times$$

$$\times E_0(r, \xi, \varkappa(t+h-\tau))\, d\xi + \int_{BQ+PA} \Big[\xi\Theta(\xi, \tau)\, E_0(r, \xi, \varkappa(t+$$

$$+h-\tau))\, d\xi + \varkappa\xi\left(E_0(r, \xi, \varkappa(t+h-\tau))\frac{\partial\Theta}{\partial\xi} - \Theta\frac{\partial E_0}{\partial\xi}\right) d\tau\Big].$$

As h $\rightarrow$ 0,

$$I_0\left(\frac{r\xi}{2\varkappa h}\right) = \exp\left(\frac{r\xi}{2\varkappa h}\right)(1+0(h)) \Big/ \sqrt{\frac{\pi r\xi}{\varkappa h}}.$$

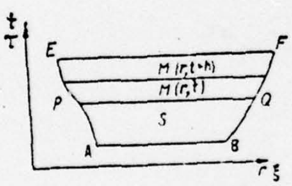

Fig. 14.

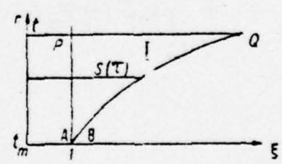

Fig. 15.

Taking this into account, we obtain

$$\int_{PQ} \xi\Theta(\xi, t) \frac{\exp\left(-\frac{r^2+\xi^2}{4\varkappa h}\right) I_0\left(\frac{r\xi}{2\varkappa h}\right)}{2\varkappa h} d\xi \xrightarrow[h\to 0]{}$$

$$\to \int_{PQ} \xi\Theta(\xi, t) \frac{\exp\left(-\frac{(r-\xi)^2}{2\varkappa h}\right)}{2\sqrt{\varkappa h\pi r\xi}} d\xi =$$

$$= \frac{1}{r}\int_{PQ} \Theta(\xi, t)\sqrt{\xi}\, \frac{\exp\left(-\frac{(r-\xi)^2}{4\varkappa h}\right)}{2\sqrt{\pi\varkappa h}} d\xi \xrightarrow[h\to 0]{} \Theta(r, t).$$

The last limit is substantiated in [86]. Hence

$$\Theta(r, t) = \int\limits_{AB}^{t} \xi\Theta(\xi, t)\, E_0(r, \xi, \varkappa(t-\tau))\, d\xi +$$
$$+ \int\limits_{BQ+PA}^{t} \left[\xi\Theta(\xi, t)\, E_0(r, \xi, \varkappa(t-\tau))\, d\xi + \varkappa\xi\left(E_0 \frac{\partial\Theta}{\partial\xi} - \Theta\frac{\partial E^0}{\partial\xi}\right) d\tau\right]. \tag{A.2}$$

Let us examine the application of the general relation (A.2) to problem (2.72)-(2.81). For a thawed zone region PABQ acquires the form shown in Fig. 15. Relation (A.2) for the indicated region is written as follows (here and below the index z is omitted):

$$\Theta_1(r, t) = \int\limits_{BQ} \xi\Theta_1(\xi, \tau)\, E_0(r, \xi, \varkappa_1(t-\tau))\, d\xi +$$
$$+ \int\limits_{PA} \varkappa_1\xi\left(E_0\frac{\partial\Theta_1}{\partial\xi} - \Theta_1\frac{\partial E_0}{\partial\xi}\right) d\tau + \int\limits_{BQ} \varkappa_1\xi\left(E_0\frac{\partial\Theta_1}{\partial\xi} - \Theta_1\frac{\partial E_0}{\partial\xi}\right) d\tau =$$
$$= \int\limits_{t}^{t_m} \left[\varkappa_1(E_0(r, 1, \varkappa_1(t-\tau)))\frac{\partial\Theta_1}{\partial\xi}\bigg|_{\xi=1} - f(\tau)\frac{\partial E_0}{\partial\xi}\bigg|_{\xi=1}\right] d\tau + \tag{A.3}$$
$$+ \int\limits_{t_m}^{t} \varkappa_1 S(\tau)\, E_0(r, S(\tau), \varkappa_1(t-\tau)) \cdot \frac{\partial\Theta_1}{\partial\xi}\bigg|_{\xi=S(\tau)} d\tau.$$

Here

$$f(t) = \Theta_1(1, t). \tag{A.4}$$

For the frozen region of problem (2.72)-(2.81) the points B and Q of region PABQ are removed to infinity. In order to show that it is possible to apply relation (A.2) to a region of this form, let us

examine the auxiliary region S_R (Fig. 16). Applying relation (A.2) to this region, we obtain

$$\Theta_R(r, t) = \int_1^R \xi\Theta_R(\xi, t_m) E_0(r, \xi, \varkappa_2(t - t_m))\, d\xi +$$
$$+ \int_t^{t_m} \varkappa_2 S(\tau) E_0(r, S(\tau), \varkappa_2(t - \tau)) \frac{\partial\Theta_R}{\partial\xi}\bigg|_{\xi=S(\tau)} d\tau + \tag{A.5}$$
$$+ \int_{t_m}^{t} \varkappa_2 R\left(E_0 \frac{\partial\Theta_R}{\partial\xi}\bigg|_{\xi=R} - \Theta_R(R, r)\frac{\partial E_0}{\partial\xi}\bigg|_{\xi=R}\right) d\tau.$$

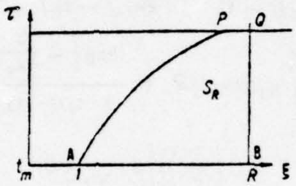

Fig. 16.

Let us examine the nature of the change in this function as R

$$\int_{t_m}^{t} \varkappa_2 R E_0(r, R, \varkappa_2(t - \tau)) \frac{\partial\Theta_R}{\partial\xi}\bigg|_{\xi=R} d\tau =$$
$$= \int_{t_m}^{t} \varkappa_2 R \frac{\exp\left(-\frac{r^2 + R^2}{4\varkappa_2(t-\tau)}\right)}{2\varkappa_2(t-\tau)} I_0\left(\frac{rR}{2\varkappa_2(t-\tau)}\right)\frac{\partial\Theta_R}{\partial\xi}\bigg|_{\xi=R} d\tau =$$
$$= \int_{t_m}^{t} \varkappa_2 R \frac{\exp\left(-\frac{r^2 + R^2}{4\varkappa_2(t-\tau)}\right)}{2\varkappa_2(t-\tau)} \cdot \frac{\exp\left(\frac{rR}{2\varkappa_2(t-\tau)}\right)\left(1 + 0\left(\frac{1}{R}\right)\right)}{\sqrt{\frac{\pi r R}{\varkappa_2(t-\tau)}}} \times$$
$$\times \frac{\partial\Theta_R}{\partial\xi}\bigg|_{\xi=R} d\tau = \int_{t_m}^{t} \varkappa_2 \sqrt{R} \frac{\exp\left(-\frac{(r-R)^2}{4\varkappa_2(t-\tau)}\right)\left(1 + 0\left(\frac{1}{R}\right)\right)}{2\sqrt{\pi\varkappa_2(t-\tau) r}} \times$$
$$\times \frac{\partial\Theta_R}{\partial\xi}\bigg|_{\xi=R} d\tau \xrightarrow[R\to\infty]{} 0.$$

This takes place for any $t > t_m$, since $\left.\frac{\partial \Theta_R}{\partial \xi}\right|_{\xi = R} \to 0$ as $R \to 0$ (the perturbations of the temperature field at infinity die down) and the exponential factor goes to $-\infty$ as $R \to \infty$.

We note that this integral is improper, with a removable singularity at the point $\tau = t$. As $\tau \to t$ the integrand goes to zero:

$$\int_{t_m}^{t} \varkappa_2 R \Theta_R (R, \tau) \left.\frac{\partial E_0}{\partial \xi}\right|_{\xi = R} d\tau = \int_{t_m}^{t} \varkappa_2 R \Theta_R (R, \tau)$$

$$\times \left[\frac{\exp\left(-\frac{r^2 + R^2}{4\varkappa_2 (t-\tau)}\right)}{2\varkappa_2 (t-\tau)} \left(-\frac{R}{2\varkappa_2 (t-\tau)} I_0\left(\frac{rR}{2\varkappa_2 (t-\tau)}\right) + \frac{r}{2\varkappa_2 (t-\tau)} \times \right.\right.$$

$$\left.\left. \times I_0'\left(\frac{rR}{2\varkappa_2 (t-R)}\right)\right)\right] d\tau = \int_{t_m}^{t} \varkappa_2 R \Theta_R (R, \tau) \frac{\exp\left(-\frac{(r-R)^2}{4\varkappa_2 (t-\tau)}\right)}{4 (\varkappa_2 (t-\tau))^{3/2} \sqrt{\pi R r}} \times$$

$$\times \left[(r - R) \left(1 + 0\left(\frac{1}{R}\right)\right)\right] d\tau \xrightarrow[R \to \infty]{} 0.$$

The region PABQ for the frozen region may be obtained from region S_R for $R \to \infty$. Hence

$$\Theta_2 (r, t) = \int_1^{\infty} \xi \Theta_2 (\xi, t_m) E_0 (r, \xi, \varkappa_2 (t - t_m)) d\xi - \tag{A.6}$$
$$- \int_{t_m}^{t} \varkappa_2 S(\tau) E_0 (r, S(\tau), \varkappa_2 (t-\tau)) \left.\frac{\partial \Theta_2}{\partial \xi}\right|_{\xi = S(\tau)} d\tau.$$

Introducing the notation

$$v_0 (t) \equiv \left.\frac{\partial \Theta_1}{\partial r}\right|_{r=1}, \quad v_1 (t) \equiv \left.\frac{\partial \Theta_1}{\partial r}\right|_{r=S}, \quad v_2 (t) \equiv \left.\frac{\partial \Theta_2}{\partial r}\right|_{r=S},$$

we obtain

$$\Theta_1(r,t) = \int_{t_m}^{t} \varkappa_1 S(\tau) E_0(r, S(\tau), \varkappa_1(t-\tau)) v_1(\tau)\, d\tau -$$
$$- \int_{t_m}^{t} \varkappa_1 E_0(r, 1, \varkappa_1(t-\tau)) v_0(\tau)\, d\tau + \int_{t_m}^{t} \varkappa_1 f(\tau) \left.\frac{\partial E_0}{\partial \xi}\right|_{\xi=1} d\tau =$$
$$= S_1 - S_2 + S_3. \tag{A.7}$$

$$\Theta_2(r,t) = \int^{\infty} \xi \Theta_0(\xi) E_0(r, \xi, \varkappa_2(t-t_m))\, d\xi -$$
$$- \int_{t_m}^{t} \varkappa_2 S(\tau) E_0(r, S(\tau), \varkappa_2(t-\tau)) v_2(\tau)\, d\tau = S_4 - S_5. \tag{A.8}$$

Let us verify the possibility of differentiating the improper integrals S_1 through S_5 with respect to the parameter r. Integral S_1 converges at any point in the interval $1 \leq r \leq S(t)$, and the result of formal differentiation of this integral under the integral sign converges uniformly on any interval $1 \leq r_1 \leq r \leq r_2 < S(t)$. Integrals S_2 and S_3 converge at any point in the interval $1 \leq r \leq S(t)$, and the results of formal differentiation of these integrals under the integral sign converge uniformly on any interval $1 \leq r_1 \leq r \leq \leq r_2 \leq S(t)$. Integral S_4 converges, and the result of formal differentiation of this integral under the integral sign converges uniformly on any interval $S(t) < r_1 \leq r \leq r_2 < \infty$. Integral S_5 converges, and the result of formal differentiation of this integral under the integral sign converges uniformly on any interval $S(t) <$

$< r_1 \leq r \leq r_2 < \infty$.

Thus, on any interval $1 < r_1 \leq r \leq r_2 < S(t)$,

$$\frac{\partial\Theta_1}{\partial r} = \int_{t_m}^{t} \varkappa_1 S(\tau) v_1(\tau) \frac{\partial}{\partial r} E_0(r, S(\tau), \varkappa_1(t-\tau))\, d\tau -$$

$$- \int_{t_m}^{t} \varkappa_1 v_0(\tau) \frac{\partial}{\partial r} E_0(r, 1, \varkappa_1(t-\tau))\, d\tau + \int_{t_m}^{t} \varkappa_1 f(\tau) \frac{\partial^2 E_0}{\partial r \partial \xi}\, d\tau.$$

Taking into account the fact that

$$\varkappa_1 \frac{\partial^2 E_0}{\partial r \partial \xi} = \frac{\partial E_1}{\partial \tau},$$

the last integral may be transformed as follows:

$$\int_{t_m}^{t} \varkappa_1 f(\tau) \frac{\partial E_1}{\partial \tau}\, d\tau = \int_{t_m}^{t} f(\tau)\, d E_1 = f(t) \cdot E_1(r, 1, 0) -$$

$$- f(t_m) E_1(r, 1, \varkappa_1(t - t_m)) - \int_{t_m}^{t} E_1(r, 1, \varkappa_1(t-\tau)) \dot{f}(\tau)\, d\tau.$$

Let us show that the first two terms are equal to zero:

$$E_1(r, 1, 0) = \lim_{t \to 0} \frac{\exp\left(-\frac{r^2+1}{4\varkappa_1 t}\right)}{2\varkappa_1 t} I_1\left(\frac{r}{2\varkappa_1 t}\right) =$$

$$= \lim_{t \to 0} \frac{\exp\left(-\frac{r^2+1}{4\varkappa_1 t}\right) \exp\left(\frac{r}{2\varkappa_1 t}\right)(1 + 0(t))}{2\varkappa_1 t \sqrt{2\pi \frac{r}{2\varkappa_1 t}}} =$$

$$= \lim_{t \to 0} \frac{\exp\left(-\frac{(r-1)^2}{4\varkappa_1 t}\right)(1 + 0(t))}{(2\varkappa_1 t)^{1/2} \sqrt{2\pi r}} = 0 \quad \text{при } r \in [r_1, r_2].$$

On the other hand, $f(t_m) = 0$, since this is the temperature at the wall at the moment thawing begins. We finally obtain for $r \in [r_1, r_2]$

$$\frac{\partial \Theta_1}{\partial r} = \int_{t_m}^{t} \varkappa_1 S(\tau) v_1(\tau) \frac{\partial}{\partial r} E_0(r, S(\tau), \varkappa_1(t-\tau))\,d\tau - \int_{t_m}^{t} \varkappa_1 v_0(\tau) \frac{\partial}{\partial r} E_0(r, 1, \varkappa_1(t-\tau))\,d\tau - \int_{t_m}^{t} E_1(r, 1, \varkappa_1(t-\tau)) \dot{f}(\tau)\,d\tau. \tag{A.9}$$

Differentiating (A.8) with respect to r on any interval $S(t) < r_1 \leq r \leq r_2 < \infty$, we obtain

$$\frac{\partial \Theta_2}{\partial r} = \int_{1}^{\infty} \xi \Theta_0(\xi) \frac{\partial}{\partial r} E_0(r, \xi, \varkappa_2(t-t_m))\,d\xi - \int_{t_m}^{t} \varkappa_2 S(\tau) v_2(\tau) \frac{\partial}{\partial r} E_0(r, S(\tau), \varkappa_2(t-\tau))\,d\tau.$$

Taking into account that

$$\frac{\partial E_0}{\partial r} = -\frac{\partial E_1}{\partial \xi} - \frac{E_1}{\xi},$$

let us transform the first integral in the following manner:

$$I = \int_{1}^{\infty} \xi \Theta_0(\xi) \frac{\partial}{\partial r} E_0(r, \xi, \varkappa_2(t-t_m))\,d\xi =$$

$$= \int_{1}^{\infty} \xi \Theta_0(\xi) \frac{\partial E_1}{\partial \xi}\,d\xi - \int_{1}^{\infty} \Theta_0(\xi) E_1\,d\xi = \xi \Theta_0(\xi) E_1 \Big|_1^{\infty} +$$

$$+ \int_{1}^{\infty} E_1\,d(\xi \Theta_0(\xi)) - \int_{1}^{\infty} \Theta_0(\xi) E_1\,d\xi.$$

Substituting the values of the limits of integration into the first term and taking into account that

$$\lim_{\xi \to \infty} \xi \Theta_0(\xi) E_1(r, \xi, \varkappa_2(t-t_m)) = 0$$

and

$$\Theta_0(1) = 0,$$

we obtain

$$I = \int_1^\infty E_1(\Theta_0(\xi) + \xi\Theta_0'(\xi))\, d\xi - \int_1^\infty \Theta_0(\xi) E_1 d\xi =$$

$$= \int_1^\infty E_1(r, \xi, \varkappa_2, (t-t_m))\, \xi\Theta_0'(\xi)\, d\xi.$$

Hence for $r \in [r_1, r_2]$ we have

$$\frac{\partial\Theta_2^*}{\partial r} = \int_1^\infty E_1(r, \xi, \varkappa_2(t-t_m))\, \xi\Theta_0'(\xi)\, d\xi - \tag{A.10}$$
$$- \int_{t_m}^{t} \varkappa_2 S(\tau) v_2(\tau) \frac{\partial}{\partial r} E_0(r, S(\tau), \varkappa_2(t-\tau))\, d\tau.$$

Let us consider

$$\lim_{r \to 1} \frac{\partial\Theta_1}{\partial r}.$$

Integral S_1 in expression (A.7) is a differentiable function of the parameter r on any interval $1 \leq r_1 \leq r \leq r_2 < S(t)$. Hence

$$\lim_{r \to 1} \frac{\partial S_1}{\partial r} = \left.\frac{\partial S_1}{\partial r}\right|_{r=1}.$$

Let us consider integral S_2. Using the integral representation of the Bessel function of an imaginary argument, we transform this integral in the following manner:

$$S_2 = \int_{t_m}^{t} \varkappa_1 \left[v_0(\tau) \frac{\exp\left(-\frac{r^2+1}{4\varkappa_1(t-\tau)}\right)}{2\varkappa_1(t-\tau)} \frac{1}{\pi} \int_0^{\pi} \operatorname{ch}\left(\frac{r\cos\Theta}{2\varkappa_1(t-\tau)}\right) d\Theta \right] d\tau =$$

$$= \int_{t_m}^{t} \left[\frac{v_0(\tau)}{2\pi(t-\tau)} \int_0^{\pi} \frac{1}{2} \exp\left(-\frac{r^2+1}{4\varkappa_1(t-\tau)}\right) \left(\exp\left(\frac{r\cos\Theta}{2\varkappa_1(t-\tau)}\right) + \right.\right.$$

$$\left.\left. + \exp\left(-\frac{r\cos\Theta}{2\varkappa_1(t-\tau)}\right)\right) d\Theta \right] d\tau = \int_{t_m}^{t} \left[\frac{v_0(\tau)}{2\pi(t-\tau)} \int_0^{\pi} \frac{1}{2} \left(\exp\left(-\frac{r^2-2r\cos\Theta+1}{4\varkappa_1(t-\tau)}\right) + \right.\right.$$

$$\left.\left. + \exp\left(-\frac{r^2+2r\cos\Theta+1}{4\varkappa_1(t-\tau)}\right)\right) d\Theta \right] d\tau.$$

We have:

$$\int_0^{\pi} \exp\left(-\frac{r^2+2r\cos\Theta+1}{4\varkappa_1(t-\tau)}\right) d\Theta = \int_0^{2\pi} \exp\left(-\frac{r^2+2r\cos(\varphi-\pi)+1}{4\varkappa_1(t-\tau)}\right) d\varphi$$

$$= \int_{\pi}^{2\pi} \exp\left(-\frac{r^2-2r\cos\varphi+1}{4\varkappa_1(t-\tau)}\right) d\varphi.$$

From this we obtain (Fig. 17)

$$S_2 = \int_{t_m}^{t} \left[\frac{v_0(t)}{4\pi(t-\tau)} \left(\int_0^{\pi} \exp\left(-\frac{r^2-2r\cos\Theta+1}{4\varkappa_1(t-\tau)}\right) d\Theta + \right.\right.$$

$$\left.\left. + \int_{\pi}^{2\pi} \exp\left(-\frac{r^2-2r\cos\Theta+1}{4\varkappa_1(t-\tau)}\right) d\Theta \right) \right] d\tau = \int_{t_m}^{t} \frac{v_0(t)}{4\pi(t-\tau)} \times$$

$$\times \int_0^{2\pi} \exp\left(-\frac{r^2-2r\cos\Theta+1}{4\varkappa_1(t-\tau)}\right) d\Theta d\tau = \int_{t_m}^{t} \frac{v_0(\tau)}{4\pi(t-\tau)} \times$$

$$\times \int_0^{2\pi} \exp\left(-\frac{\rho^2}{4\varkappa_1(t-\tau)}\right) d\Theta d\tau = \int_{t_m}^{t} \int_{\gamma} \frac{v_0(\tau)\exp\left(-\frac{\rho^2}{4\varkappa_1(t-\tau)}\right)}{4\pi(t-\tau)} d\gamma d\tau.$$

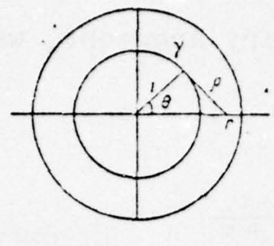

Fig. 17.

Here γ is the unit circle. Thus, integral S_2 is in the form of the two-dimensional thermal potential of a simple layer [87].

Using the theorem regarding the discontinuities of the normal derivative of the thermal potential of a simple layer [88], we obtain

$$\lim_{r \to 1} \frac{\partial S_2}{\partial r} = -\frac{\varphi_0(t)}{2} + \left.\frac{\partial S_2}{\partial r}\right|_{r=1}.$$

The last integral in expression (A.9) uniformly converges in r on an interval which includes the point r = 1. Let us show this. Let us examine the behavior of the integrand as $\tau \to t$.

$$\dot{f}(\tau)\, E_1(r, 1, \varkappa(t-\tau)) = \dot{f}(\tau) \frac{\exp\left(-\frac{r^2+1}{4\varkappa_1(t-\tau)}\right)}{2\varkappa_1(t-\tau)} \times$$

$$I_0\left(\frac{r}{2\varkappa_1(t-\tau)}\right) \xrightarrow[\tau \to t]{} \dot{f}(\tau) \frac{\exp\left(-\frac{r^2+1}{4\varkappa_1(t-\tau)}\right)\exp\left(\frac{r}{2\varkappa_1(t-\tau)}\right)}{2\varkappa_1(t-\tau)\sqrt{2\pi\frac{r}{2\varkappa_1(t-\tau)}}} \times$$

$$\times\left(1+0\left(\frac{2\varkappa_1(t-\tau)}{r}\right)\right) = \dot{f}(\tau) \frac{\exp\left(-\frac{(r-1)^2}{4\varkappa_1(t-\tau)}\right)}{\sqrt{4\pi r\varkappa_1(t-\tau)}}\left(1+0\left(\frac{2\varkappa_1(t-\tau)}{r}\right)\right) \leqslant$$

$$\leqslant \dot{f}(\tau)\frac{1}{\sqrt{4\pi r\varkappa_1(t-\tau)}} \quad \text{при } r \in [1, r_2].$$

It is obvious that the integral of the last function (majorante) converges ($\dot{f}(\tau)$ is bounded), and consequently the integral in question converges uniformly on any interval $[1, r_2]$. Hence

$$\lim_{r \to 1} \int_{t_m}^{t} \dot{f}(\tau) E_1(r, 1, \varkappa_1(t-\tau))\, d\tau = \int_{t_m}^{t} \dot{f}(\tau) E_1(1, 1, \varkappa_1(t-\tau))\, d\tau.$$

We finally obtain

$$\lim_{r \to 1} \frac{\partial \Theta_1}{\partial r} = v_0(t) = \int_{t_m}^{t} \varkappa_1 S(\tau) v_1(\tau) \frac{\partial}{\partial r} E_0(1, S(\tau), \varkappa_1(t-\tau))\, d\tau +$$
$$+ \frac{v_0(t)}{2} - \int_{t_m}^{t} \varkappa_1 v_0(\tau) \frac{\partial}{\partial r} E_0(1, 1, \varkappa_1(t-\tau))\, d\tau -$$
$$- \int_{t_m}^{t} \dot{f}(\tau) E_1(1, 1, \varkappa_1(t-\tau))\, d\tau.$$

Let us examine

$$\lim \frac{\partial \Theta_1}{\partial r} \qquad \text{as} \qquad r \to S(t).$$

Analogous to what was done above, one may show that the integral in (A.7) may be put into the form (Fig. 18)

$$S_1 = \int_{t_m}^{t} \int_{\gamma} v_1(\tau) \frac{\exp\left(-\frac{\rho^2}{4\varkappa_1(t-\tau)}\right)}{4\pi(t-\tau)}\, d\gamma d\tau.$$

According to the theorem regarding the discontinuity of the normal derivative of the thermal potential of a simple layer we

obtain

$$\lim_{r\to S(t)} \frac{\partial S_1}{\partial r} = \frac{v_1(t)}{2} + \left.\frac{\partial S_1}{\partial r}\right|_{r=S(t)}.$$

It is easy to show that the second and third integrals in expression (A.9) uniformly converge in r on any interval $1 < r_1 \leq \leq r \leq S(t)$, and consequently are continuous in r at the point $r = S(t)$. Taking this into account we obtain

$$\lim_{r\to S(t)} \frac{\partial \Theta_1}{\partial r} = v_1(e) = \frac{v_1(t)}{2} + \int_{t_m}^{t} \varkappa_1 S(\tau)\, v_1(\tau) \cdot \frac{\partial}{\partial r} E_0 \times$$

$$\times (S(t), S(\tau), \varkappa_1(t-\tau))\, d\tau - \int_{t_m}^{t} \varkappa_1 v_0(\tau) \frac{\partial}{\partial r} E_0 \times$$

$$\times (S(t), 1, \varkappa_1(t-\tau))\, d\tau - \int_{t_m}^{t} f(\tau)\, E_1(S(t), 1, \varkappa_1(t-\tau))\, d\tau.$$

Consider

$$\lim_{r\to S(t)} \frac{\partial \Theta_2}{\partial r}.$$

As was done above, one may show that integral S_5 in expression (A.8) may be put into the form of the thermal potential of a simple layer (Fig. 19)

$$S_5 = \int_{t_m}^{t} \int_{\gamma} v_2(\tau) \frac{\exp\left(-\frac{\rho^2}{4\varkappa_2(t-\tau)}\right)}{4\pi(t-\tau)}\, d\gamma\, d\tau.$$

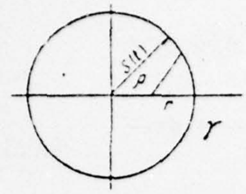

Fig. 18.

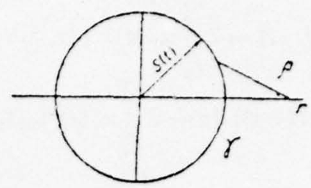

Fig. 19.

Hence

$$\lim_{r \to S(t)} \frac{\partial S_5}{\partial r} = -\frac{v_2(t)}{2} + \left.\frac{\partial S_5}{\partial r}\right|_{r=S(t)}.$$

The first integral in expression (A.10) uniformly converges in r on any interval $S(t) \leq r \leq r_2 < \infty$, and consequently is continuous in r at the point r = S(t). Hence we finally obtain

$$\lim_{r \to S(t)} \frac{\partial \Theta_2}{\partial r} = v_2(t) = \int_1^{\infty} E_1(S(t), \xi, \varkappa_2(t-t_m))\, \xi \Theta_0'(\xi)\, d\xi + \frac{v_2(t)}{2} -$$
$$- \int_{t_m}^{t} \varkappa_2 S(\tau)\, v_2(\tau) \frac{\partial}{\partial r} E_0(S(t), S(\tau), \varkappa_2(t-\tau))\, d\tau.$$

Thus, problem (2.72)-(2.80) reduces to a system of integral equations of the Volterra type:

$$v_0(t, z) = 2\int_{t_m}^{t} \varkappa_1(z)\, S(\tau, z)\, v_1(\tau, z) \frac{\partial}{\partial r} E_0(1, S(t-\tau), \varkappa_1(z)(t-\tau))\, d\tau -$$
$$- 2\int_{t_m}^{t} \varkappa_1(z)\, v_0(\tau, z) \frac{\partial}{\partial r} E_0(1, 1, \varkappa_1(z)(t-\tau))\, d\tau - 2\int_{t_m}^{t} f_\tau^{\cdot}(\tau, z) \times \qquad \text{(A.11)}$$
$$\times E_1(1, 1, \varkappa_1(z)(t-\tau))\, d\tau;$$

$$v_1(t, z) = 2\int_{t_m}^{t} \varkappa_1(z)\cdot S(\tau, z)\, v_1(\tau, z)\, \frac{\partial}{\partial r} E_0(S(t, z), S(\tau, z), \varkappa_1(z)\times$$

$$\times (t-\tau))\, d\tau - 2\int_{t_m}^{t} \varkappa_2(z)\, v_0(\tau, z)\, \frac{\partial}{\partial r} E_0(S(t, z), 1, \varkappa_1(z)(t-\tau))\, d\tau - \tag{A.12}$$

$$-2\int_{t_m}^{t} \dot{f}_\tau(\tau, z)\, E_1(S(t, z), 1, \varkappa_1(z)(t-\tau))\, d\tau;$$

$$v_2(t, z) = 2\int_{1}^{\infty} E_1(S(t, z), \xi, \varkappa_2(z)(t-t_m))\, \xi\cdot\Theta_0'(\xi, z)\, d\xi -$$

$$-2\int_{t_m}^{t} \varkappa_2(z)\cdot S(\tau, z)\cdot v_2(\tau, z)\, \frac{\partial}{\partial r} E_0(S(t, z), S(\tau, z), \varkappa_2(z)(t-\tau))\, d\tau. \tag{A.13}$$

Here it is necessary to add Stefan's condition (2.81), which, taking into account the adopted notation, may be written in the form

$$S(t, z) = 1 + \int_{t_m}^{t} [-\lambda_1(z)\, v_1(\tau, z) + \lambda_2(z)\, v_2(\tau, z)]\, d\tau. \tag{A.14}$$

and the expressior for the temperature of the wall of the borehole, according to (2.74):

$$f(t, z) = \frac{v_0(t, z)}{\alpha} + T(t, z). \tag{A.15}$$

The system is completed by the equation for the temperature of the gas, which may be obtained from (2.83), if one neglects, as is usually done for gas, the inertia term

$$T(t, z) = 1 + \int_0^z \frac{v_0(t, z) - C_1}{C_2} dz. \tag{A.16}$$

System of equations (A.11)-(A.16) was solved using the method of successive approximations on a BESM-3M computer. In order to do this, a program was written in the ALGOL language for the α-translator. Let us clarify the basic principles on which the program works. In each horizontal section $z = z_i$, beginning with the section $z = 0$, plane problem (A.11)-(A.15) was solved. Using the function v_0 obtained from this solution, from equation (A.16) the function T was determined for the subsequent section $z_{i+1} = z_i + \Delta z$. This function was substituted into system (A.11)-(A.15), and the values of all the desired functions for this section were determined. The procedure was then repeated. In order to solve plane problem (A.11)-(A.15), n points on the time axis were singled out. The coordinate of the mth point was denoted by D[m], where m = 1, 2, ..., n. The values of the unknown functions v_0, v_1, v_2, s, f, and T at these points were denoted respectively by v0j 1 [m, p], vljl [m, p], v2jl [m, p], sj [m, p], fj [m, p], Tgj[m, p] (previous approximation) and v0j[m, p], vlj[m, p], v2j[m, p], sjl[m, p], fjl[m, p] (next approximation).

In order to construct the function over the entire axis from its values at n points, the interpolation procedure w was introduced, which to each of the discrete functions v0jl [m, p], vljl [m, p], ..., determined at the n points, put in correspondence the continuous functions w[y, v0jl] ..., determined in the following way. At the reference points D[m] the values of the functions w[y, ...] coincided with the values of the corresponding discrete functions, and at intermediate points the values were determined by linear interpolation. For example, the function w [y, v0jl] is a continuous function, defined in the following manner: for $y \in$ [D[m], D[m+1]],

$$w[y, v0jl] = (v0jl[m+1, p] \times (y - D[m]) + \\ + v0jl[m, p] \times (D[m+1] - y)) / (D[m+1] - D[m]).$$

Using the functions w[y, ...], for each mth point were constructed the functions z0, zl, z2, which are the integrands in the integrals over the time axis in equations (A.11), (A.12), and (A.13) respectively, and z3, for the volume integral in equation (A.13). The functions v0j[m, p], vlj[m, p] were determined by integrating the functions z0 and zl respectively within the limits D[l] to D[m]. The function v2j [m, p] was determined analogously, with this difference: to the integral of the function z2 within the limits D[l] to D[m] was added the volume integral of the function z3.

The time integrals in equations (A.11)-(A.13) are improper, with a singularity at the upper limit of the form $1/\sqrt{t - \tau}$.

Correspondingly, the functions z0, z1, z2 also have at the points D[m] singularities of this form. Taking this into account, the region of integration was broken up into two intervals: [D[1], D[m] - ε1[m]] and [D[m] - ε1[m], D[m]], where ε1[m] is a value that is so small that in the last interval the integrand may, with a sufficient degree of accuracy, be approximated by a function of the form $\frac{a}{\sqrt{t-\tau}} + b$.

In the first interval the integral was calculated using the standard procedure "simps;" the integral in the second interval was calculated analytically from the approximating function $\frac{a}{\sqrt{t-\tau}} + b$, the coefficients in which were determined from the values of the integrand at the points D[m] - ε1[m] and D[m] - - ε1[m]/2.

Thus, given some first approximation, i.e. a set of discrete functions v0j1[m, p], v1j1[m, p], ..., successively, for each m, we obtain using these functions a new set of functions v0j[m, p], v1j[m, p], ... The second approximation is taken in the form

$$\frac{v0j1[m,p] + v0j[m,p]}{2},$$

$$\frac{v1j1[m,p] + v1j[m,p]}{2}, \ldots$$

If the next approximationsdiffered from the previous ones by less than some small value ε2, the solution at the given point was taken to be equal to the latter approximation. Otherwise, the iteration process was continued. It is necessary to note that convergence of the iteration process was assured everywhere, except

for a section of the time axis directly adjoining the coordinate origin. In this section the form of the unknown functions was given beforehand in accordance with a well-known approximate solution. As an example, Fig. 20 shows the results of solving the plane problem for $\Theta_F = -0.1$. The vector D was assigned in the form

$$D = \begin{Bmatrix} 0{,}0 \\ 0{,}5 \\ 1{,}0 \\ 2{,}0 \\ 4{,}0 \\ 8{,}0 \\ 12{,}0 \\ \cdots \\ 52{,}0 \end{Bmatrix}$$

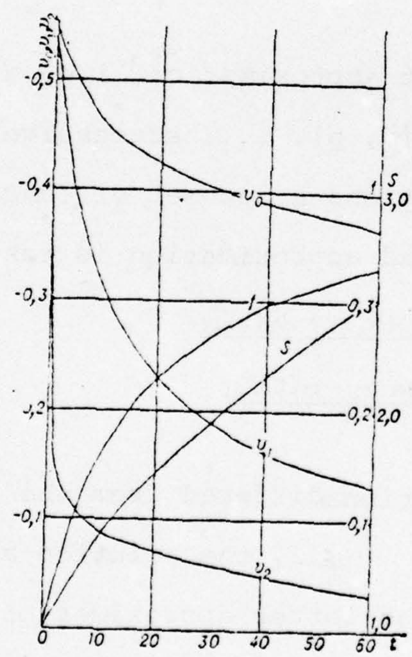

Fig. 20.

Consequently, the partition points concentrated near the coordinate origin, where the most intensive change in the unknown functions takes place. At the points D[1],D[2],D[3] the values of the functions were assigned from the approximate solution, and at the remaining points they were determined using the described iteration process.

The results of solving individual problems using this program are shown as graphs together with the results of calculations carried out using approximation methods. Comparison of these results is made in the paragraphs devoted to the corresponding approximation methods.

2. Numerical Solution of the Problem of Heat Exchange During Flushing of a Well

In this case the heat problem in ground also reduces to system of equations (A.11)-(A.14). Equation (A.15) preserves its form, with this sole difference: in place of T in this equation it is necessary to substitute T_2, the temperature of the flushing fluid in the annular space.

The basic difference will consist in the conditions of heat balance in the well, which in our case are described by equations (3.89)-(3.90). Let us transform these equations in the following manner. Differentiating equation (3.90) with respect to z, we have:

$$\frac{\partial^2 T_1}{\partial z^2} = -\varkappa_1\left(\frac{\partial T_1}{\partial z} - \frac{\partial T_2}{\partial z}\right). \qquad (A.17)$$

Equation (3.89), taking into account (3.90), may be written in the form

$$\frac{\partial T_2}{\partial z} = -\alpha_3 v_0 + \frac{\partial T_1}{\partial z}. \tag{A.18}$$

Here

$$\alpha_3 = \frac{2\pi L\bar{\lambda}_1}{cG}, \quad v_0 = \left.\frac{\partial \Theta_1}{\partial r}\right|_{r=1}.$$

Substituting (A.18) into (A.17), we obtain

$$\frac{\partial^2 T}{\partial z^2} = -\alpha_1 \alpha_3 v_0. \tag{A.19}$$

Thus, adding to system of equations (A.11)-(A.14) equations (A.18) and (A.19), and also the equation

$$f = \frac{v_0}{\alpha} + T_2, \tag{A.20}$$

we obtain a complete system of equations for determining the unknown functions. This system was solved on a BESM-3M computer. The program was written in ALGOL for the α-translator.

The fundamental ideas on which the program worked are as follows. The entire stratum is broken up into q horizontal sections. On the time axis n points were singled out; the coordinate of the mth point was denoted by D[m], where m = 1, 2, ..., n. For each moment in

time, beginning with $m = 2$, system (A.11)-(A.14), (A.20) was solved for all the horizontal sections using the method presented above. The obtained function v_0 was used for determining the functions T_1 and T_2 for the given moment in time from equations (A.18)-(A.19). The function T_2 was substituted into equation (A.20), and system (A.11)-(A.14), (A.20) was again solved for the next moment in time.

APPENDIX 2

Program for Solving System of Integrodifferential Equations (A.11)-(A.16) Using the Programming Language ALGOL-60 for the α-Translator of a BESM-3M Computer

```
begin real T, α, d, f, a1, a2, ε2, ε3, ε4,
tm, v0, v1, v2, b1, b2, α2, Rt, Rm;
integer m, n, p, q, i, j, N; read (n, q); begin,
array v0j, v1j, v2j, fj, sj, Tgj, v0j1, v1j1, v2j1,
fj1, sj1[1 : n, 1 : q], ϰ1, ϰ2, λ1, λ2, b, F0[1 : q],
ε1, D[1 : n];
real procedure I0(y);
value y; real y;
begin if abs(y) < 10 then go to Nt else
Mt: begin real pe, sum; integer r;
r:=1; sum:=0; pe:=−1/(8×y);
it: sum:=sum+(−1)↑r×pe; r:=r+1;
pe:=pe×(−(2×r−1)↑2/(8×r×y));
if abs(pe) > ⏨−4 then go to it else
I0:=(1+sum)/sqrt(abs(y)×6.2832);
go to P end Mt;
Nt: begin real pe, sum; integer r;
r:=1; sum:=0; pe:=(y/2)↑2;
Lt: sum:=sum+pe; r:=r+1;
pe:=pe×(y/(2×r))↑2;
if r<10 then go to Lt else
I0:=(1+sum)×exp(−y) end Nt; P:
end;
real procedure I1(y);
value y; real y;
begin if abs(y) < 10 then go to N1 else
M1: begin real pe, sum; integer r;
r:=1; sum:=0; pe:=3/(8×y);
iter: sum:=sum+(−1)↑r×pe; r:=r+1;
pe:=pe×(4−(2×r−1)↑2)/(8×r×y);
if abs(pe) > 10^-4 then go to iter else
I1:=(1+sum)/sqrt(abs(y)×6.2832);
go to P1 end M1;
```

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```
N1: begin real pe, sum; integer r;
r:=0; sum:=0; pe:=y/2;
L1: sum:=sum+pe; r:=r+1;
pe:=pe×(y/2)↑2/(r×(r+1));
if r<10 then go to L1 else
I1:=sum×exp(−y) end N1; P1:
end;
real procedure W(x, A); value x, A;
real x; array A; begin integer j;
j:=1; TS: if x<D[j+1] then begin W:=
(A[j+1, p]×(x−D[j])+A[j, p]×(D[j+1]−x))/
(D[j+1]−D[j]); go to RS end else
j:=j+1; go to TS; RS: end;
real function d1(x, y)=(I0(1/(2×ϰ1[p]×(x−y)))−
I1(1/(2×ϰ1[p]×(x−y))))×W(y, v0j1);
real function d2(x, y)=exp(−(W(y, sj)−1)↑2/(4×ϰ1[p]×
(x−y)))×(I0(W(y, sj)/(2×ϰ1[p]×(x−y)))−W(y, sj)×
I1(W(y, sj)/(2×ϰ1[p]×(x−y))))×W(y, sj)×W(y, v1j1);
real function dif0(x, y, ε)=(1/(2×ϰ1[p]×(x−y)↑2))×
(d1(x, y)−2×(x−y)×I1(1/(2×ϰ1[p]×(x−y)))×
×(W(y+ε,
fj)−W(y, fj))/ε−d2(x, y));
real function d3(x, y)=exp(−(W(x, sj)−1)↑2/(4×
ϰ1[p]×(x−y)))×(W(x, sj)×I0(W(x, sj)/(2×ϰ1[p]×
×(x−y)))
−I1(W(x, sj)/(2×ϰ1[p]×(x−y))))×W(y, v0j1);
real function d4(x, y)=exp(−(W(x, sj)−W(y, sj))↑2
/(4×ϰ1[p]×(x−y)))/(2×ϰ1[p]×(x−y)↑2)×
(−I0(W(x, sj)×
W(y, sj)/(2×ϰ1[p]×(x−y)))×W(x, sj)+I1(W(x, sj)×
W(y, sj)/(2×ϰ1[p]×(x−y)))×W(y, sj))×W(y, sj)×
W(y, v1j1);
real function dif1(x, y, ε)=(1/(2×ϰ1[p]×(x−y)↑2))
×d3(x, y)−exp(−(W(x, sj)−1)↑2/(4×ϰ1[p]×
(x−y)))/(ϰ1[p]×(x−y))×I1(W(x, sj)/(2×ϰ1[p]×
×(x−y)))
×(W(y+ε, fj)−W(y, fj))/ε+d4(x, y);
real function dif2(x, y)=W(y, sj)×W(y, v2j1)×
exp(−(W(x, sj)−W(y, sj))↑2/(4×ϰ2[p]×(x−y)))×
(W(x, sj)×I0(W(x, sj)×W(y, sj)/(2×ϰ2[p]×(x−y)))−
W(y, sj)×I1(W(x, sj)×W(y, sj)/(2×ϰ2[p]×(x−y))))/
(2×ϰ2[p]×(x−y)↑2);
real function z0(y)=dif0(D[m], y, 10⁻²);
real function z1(y)=dif1(D[m], y, 10⁻²);
real function z2(y)=dif2(D[m], y);
```

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```
real function F(x) == (b1×(x×(1− α2/2)+α2×x×
ln(x))+b2/x)/(ϰ2[p]×D[m]);
real function z3(y) = y×F(y)×exp(−(sj[m, p]−
y)↑2/(4×ϰ2[p]×D[m]))+I1(sj[m, p]×y/(2×
ϰ2[p]×D[m]));
read (T, α, ϰ1, ϰ2, λ1, λ2, d, F0, f, a1, a2, ε1, ε2,
fj, ε3, D, j, ε4, tm, Rm, α2, Tgj, sj, v0j1, v1j1, v2j1);
for p:=1 step 1 until q do
B0: begin Rt:=4×ϰ2[p]×(α2×ln(Rm)+1)↑2/(Rm×
(2×(α2×ln(Rm)+1)↑2−2×α2×(α2×ln(Rm)+1)+
+α2↑2×
(2−2×α2+α2↑2)/(Rm)↑2)));
b1:=α2+Rt×(Tgj[1, p]−F0[p])/(2×ϰ2[p]×Rm×(α2×
ln (Rm)+1)↑2);
b2:=α2+Rt×(Tgj[1, p]−F0[p])×((2×α2−α2↑2−2)/
Rm−α2×Rm×(1−α2+α2×ln(Rm)))/(4×ϰ2[p]×
(α2×ln(Rm)+1)↑3)+α2×(F0[p]−Tgj[1, p])/(α2×
ln (Rm)+1);
v0j[1, p]:=v0j1[1, p]; v1j[1, p]:=v1j1[1, p];
v2j[1, p]:=v2j1[1, p] end B0;
Rot: begin for p:=1 step 1 until q do
R2: begin for m:=j step 1 until n do
R4: begin v2j[j−1, p]:=v2j1[j−1, p]; v1j[j−1, p]
:=v1j1[j−1, p];
v0:=simps(z0, 0, D[m−1], ε4, ε4);
v1:=simps(z1, 0, D[m−1], ε4, ε4);
v2:=simps(z2, 0, D[m−1], ε4, ε4);
A5: v0j[m, p]:=v0+simps(z0, D[m−1], D[m]−f,
ε3, ε3)+simps(z0, D[m]−f, D[m]−ε1[m], ε3, ε3);
v0j[m, p]:=v0j[m, p]+ε1[m]/(sqrt(2)−1)×((sqrt
(2)−2)×z0(D[m]−ε1[m])+z0(D[m]−ε1[m]/2))−
fj[1, p]/(ϰ1[p]×D[m])×I1(1/(ϰ1[p]×D[m]×2));
v1j[m, p]:=v1+simps(z1, D[m−1], D[m]−f,
ε3, ε3)+simps(z1, D[m]−f, D[m]−ε1[m],
ε3, ε3); v1j[m, p]:=v1j[m, p]+ε1[m]/(
sqrt(2)−1)×((sqrt(2)−2)×z1(D[m]−ε1[m])+
z1[m]−ε1[m]/2))−fj[1, p]×exp(−(sj[m, p]−1)↑
2/(4×ϰ1[p]×D[m]))×I1(sj[m, p]/(2×ϰ1[p]×
D[m]))/(ϰ1[p]×D[m]);
fj1[m, p]:=(1/α)×v0j[m, p]+Tgj[m, p];
v2j[m, p]:=v2+simps(z2, D[m−1], D[m]−f,
ε3, ε3)+simps(z2, D[m]−f, D[m]−ε1[m],
ε3, ε3)+ε1[m]/(sqrt(2)−1)×((sqrt(2)−2)×
z2(D[m]−ε1[m])+z2(D[m]−ε1[m]/2))+simps
(z3, 1.0, Rm, ε3, ε3);
```

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```
sj1[m, p]:=sj[m-1, p]+(D[m]-D[m-1])/2×
(-λ1[p]×(v1j[m, p]+v1j[m-1, p])+λ2[p]×
(v2j[m, p]+v2j[m-1, p]));
v0j1[m, p]:=0.5×(v0j1[m, p]+v0j[m, p]);
v1j1[m, p]:=0.5×(v1j1[m, p]+v1j[m, p]);
v2j1[m, p]:=0.5×(v2j1[m, p]+v2j[m, p]);
fj[m, p]:=0.5×(fj[m, p]+fj1[m, p]);
sj[m, p]:=0.5×(sj[m, p]+sj1[m, p]);
write (v0j1[m, p], v1j1[m, p], v2j1[m, p],
Tgj[m, p], fj[m, p], sj[m, p]);
B: if abs(v1j1[m, p]-v1j[m, p])+abs(v0j1
[m, p]-v0j[m, p])>ε2 then go to As
else end;
for m:=1 step 1 until n do begin
Tgj[m, p]:=Tgj[m, p]+(a1×v0j[m, p]-a2)×d
end;
v0j[1, p]:=v1j[1, p]:=-α×Tgj[1, p];
v0j1[1, p]:=v1j1[1, p]:=v0j[1, p]; end R2
end Rot; go to Rot end end
```

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